Multivariable calculus Summary

Chapter 1 – Convergence & continuity

① **Obvious properties:**

- $x_j \in \mathbb{R}^n$, $x_j \longrightarrow x \in \mathbb{R}^n$ if $\forall \varepsilon > 0 \ \exists N \in \mathbb{N}$ s.t. $j \geq N \Rightarrow |x_j - x| < \varepsilon$
- limits are unique in $\mathbb{R}^n$

  Pf: pick $\varepsilon = \frac{1}{2}|x - \tilde{x}|$. $x_j \to x$ & $x_j \to \tilde{x}$. For $j > \max\{N_1, N_2\}$
  $|x - \tilde{x}| \leq |x - x_j| + |x_j - \tilde{x}| = 2\varepsilon$
- componentwise convergence. $x_j \longrightarrow x \iff \forall i \in \{1, \dots, n\} \ \lim_{j \to \infty} x_{ij} = x_i$

  Pf: '$\Rightarrow$' $\forall i$, $|x_{ij} - x_i| \leq |x_j - x| < \varepsilon$ as $x_j \longrightarrow x$
  '$\Leftarrow$' each component converges so $N = \max\{N_1, \dots, N_n\}$, $j > N$ $|x_i - x_{ij}| < \varepsilon$
  $$|x - x_j| = \left(\sum_{i=1}^{n} (x_i - x_{ij})^2\right)^{\frac{1}{2}} \leq (n\varepsilon^2)^{\frac{1}{2}} = \sqrt{n}\,\varepsilon$$
- $x_j \longrightarrow x$ then x_j bbd

  Pf: if $x_j \longrightarrow x \Rightarrow |x_j| \longrightarrow |x|$ [because $||x_j| - |x|| \leq |x_j - x| < \varepsilon$]
  $|x_j|, |x| \in \mathbb{R}$ so year one says $|x_j|$ bounded $\Rightarrow x_j$ bbd.
- Bolzano Weierstrass: Bbd sequence $x_j \in \mathbb{R}^n$ has a convergent subsequence x_{j_l}.

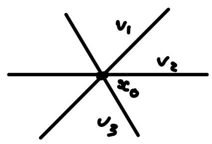
  Pf: $n=2$. (a_j, b_j) bbd in $\mathbb{R}^2 \Rightarrow |a_j|$ bbd in $\mathbb{R} \Rightarrow \exists\, a_{j_k} \longrightarrow a$
  Then b_{j_k} bbd $\Rightarrow \exists\, b_{j_{k_l}} \longrightarrow b$. $a_{j_{k_l}} \longrightarrow a$ still so $(a_{j_{k_l}}, b_{j_{k_l}}) \to (a,b)$
- sequential and ε-δ continuity equivalent.

  Pf: '$\Leftarrow$' suppose f cts at $c \in \mathbb{R}^n$ & $x_n \longrightarrow c$, write out defn, pick N s.t. $\forall n > N$,
  $|x_n - c| < \delta \Rightarrow \forall n > N \ |f(x_n) - f(c)| < \varepsilon \Rightarrow f(x_n) \longrightarrow f(c)$
  '$\Rightarrow$' Contrapositive! suppose f not cts at $c \in \mathbb{R}^n$. Pick ε s.t. $\forall \delta > 0$
  $\exists x$ s.t. $|x - c| < \delta \Rightarrow |f(x) - f(c)| \geq \varepsilon$. Pick x_n s.t. $|x_n - c| < \frac{1}{n}$, but
  $|f(x_n) - f(c)| \geq \varepsilon$. $x_n \longrightarrow c$ but $f(x_n) \not\longrightarrow f(c)$
- Continuous limit: $\lim_{x \to p} f(x) = q$ if $\forall \varepsilon > 0 \ \exists \delta > 0$ s.t. $x \in U$ $0 < |x - p| < \delta \Rightarrow |f(x) - q| < \varepsilon$
- Separate continuity: $f: \mathbb{R}^2 \longrightarrow \mathbb{R}$ separately cts at (x_0, y_0) if g^{y_0} cts at x_0 and h^{x_0} cts at y_0.
- Algebra of cts functions Pf: use real valued analogues & sequential for composition.

② **constructing multivariable cty from $\mathbb{R}$ cty:** $g: E \subset \mathbb{R} \longrightarrow \mathbb{R}$ cts at a.
Define $f: U_i \longrightarrow \mathbb{R}$ $[\forall i \in \{1, \dots, n\}]$ $U_i = \{(x_1, \dots, x_n) \in \mathbb{R}^n : x_i \in E\}$, $f(x_1, \dots, x_n) =: g(x_i)$
Then f cts on $\{(x_1, \dots, x_n) \in \mathbb{R}^n : x_i = a\}$

Don't quote explicitly. $F(x,y) = \frac{x^2 y}{x^2 + y^2}$ is cts on $\mathbb{R}^2 \setminus \{0,0\}$ $\because$ quotient of polynomials

③ **cty along lines / linear cty:** f cts along lines at x_0 if $f(x_0 + tv)$ cts at $t=0$ for every choice of $v \in \mathbb{R}^n$.
$$\lim_{t \to 0} f(x_0 + tv) = f(x_0) \quad \forall v \in \mathbb{R}^n$$



④ **Counterexamples for implications:**

Chapter 2 - topology basics & cty

(5) Open & closed sets:

- $X \subset \mathbb{R}^n$ is closed if $x_j \in X$, $x_j \longrightarrow x \in \mathbb{R}^n$, then $x \in X$
- $X \subset \mathbb{R}^n$ is open if $\forall x \in X$, $\exists \varepsilon > 0$ s.t. $\mathbb{B}(x, \varepsilon) \subset X$

Pf of equivalence: '$\Rightarrow$' contradiction. $\exists y \in X^c$, $x_j \in X$ s.t. $|x_j - y| \le \frac{1}{j}$ $\Rightarrow \lim_{j \to \infty} x_j = y$
X closed so $y \in X$ ✗ '$\Leftarrow$' again, contradiction.

- An arbitrary union of open sets is open: U_λ open $\Rightarrow \bigcup_{\lambda \in I} U_\lambda$ open

Pf: $p \in \bigcup_{\lambda \in I} U_\lambda \Rightarrow \exists \lambda^* \in I$ s.t. $p \in U_{\lambda^*}$ ← open $\Rightarrow \exists \varepsilon$ s.t.
$\mathbb{B}(p, \varepsilon) \subset U_{\lambda^*} \subset \bigcup_{\lambda \in I} U_\lambda$ ← open!

De Morgan's laws

$$\boxed{\overline{\bigcup U_i} = \bigcap \overline{U_i}}$$

$$\boxed{\overline{\bigcap U_i} = \bigcup \overline{U_i}}$$

- ε neighbourhood: $E \subset \mathbb{R}^n$. Given $\varepsilon > 0$, define ε-neighbourhood

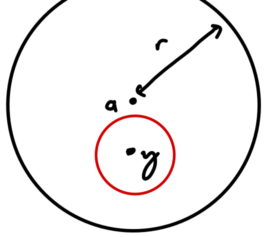
$$N(E, \varepsilon) = \bigcup_{x \in E} \mathbb{B}(x, \varepsilon)$$

- Finite intersection of open sets open: $U_1, \dots, U_m$ open $\Rightarrow \bigcap_{i=1}^{m} U_i$ open

Pf: $p \in \bigcap U_j \Rightarrow \exists \varepsilon_i > 0$ s.t. $\mathbb{B}(p, \varepsilon_i) \subset U_i$. Set $\varepsilon = \min\{\varepsilon_1, \dots, \varepsilon_m\}$
then $\mathbb{B}(p, \varepsilon) \subset \bigcap_{j=1}^{m} U_j$ ← open

For arbitrary $\cap$ closed = closed, finite $\cup$ closed is closed use demorgan.

- 'Open' ball is open



Pf: Pick $y \in \mathbb{B}(a, r)$. Set $\delta = r - |y - a|$. Claim $\mathbb{B}(y, \delta) \subset \mathbb{B}(a, r)$

Then $x \in \mathbb{B}(y, \delta)$, $|x - a| \le |x - y| + |y - a| \le \delta + |y - a| = r$

(6) Continuity by open/closed sets: Following equivalent: [proof non-examinable]

(i) $f: \mathbb{R}^n \longrightarrow \mathbb{R}^k$ cts on all of $\mathbb{R}^n$

(ii) $\forall$ open subsets V of $\mathbb{R}^k$, $f^{-1}(V)$ open

(iii) $\forall$ closed " F " , $f^{-1}(F)$ closed

(7) Continuity & sequential compactness:

limit lies in K

- $K \subset \mathbb{R}^n$ sequentially compact if for every sequence $x_j \in K$ $\exists x_{j_k} \longrightarrow x \in K$
- $X \subset \mathbb{R}^n$ bbd if $\exists M$ s.t. $\forall x \in X$ $|x| < M$
- $K \subset \mathbb{R}^n$ sequentially compact $\Leftrightarrow$ K is closed & bbd.

Pf: '$\Rightarrow$' if K sequentially compact, $x_j \in K$, $\exists x_{j_k} \longrightarrow x \in K$.
$x = \lim_{j \to \infty} x_j = \lim_{k \to \infty} x_{j_k} \in K$ so every sequence has a limit in K. Closed.

'$\Leftarrow$' Suppose K unbdd. Then $\exists x_j \in K$ s.t. $|x_j| \ge j$ $\forall j \in \mathbb{N}$. K sequentially compact.
$\exists x_{j_k} \longrightarrow x \in K \Rightarrow x_{j_k}$ bbd, $\exists M$ s.t. $|x_{j_k}| \le M$ $\forall k \in \mathbb{N}$.
$M \ge |x_{j_k}| \ge j_k \ge k$ $\forall k \in \mathbb{N}$ ✗ K bbd

Assume K closed & bbd. Bolzano Weierstrass: $\exists x_{j_k} \longrightarrow x \in K$ [closed]

- Cty preserves sequential compactness: $f(K)$ sequentially compact if K is & $f: K \longrightarrow \mathbb{R}^k$ cts
- Extreme value thm: $K \subset \mathbb{R}^n$ sequentially compact. [attains bounds]

(A) $f: K \longrightarrow \mathbb{R}$ cts. $\exists x_*, x^* \in K$ s.t. $f(x_*) \le f(x) \le f(x^*)$ $\forall x \in K$ Pf: There are sequences whose limit attains $\sup f(K)$, $\inf f(K)$.

(B) $f: K \longrightarrow \mathbb{R}^k$ cts $\exists x_*, x^* \in K$ s.t. $|f(x_*)| \le |f(x)| \le |f(x^*)|$ $\forall x \in K$

Chapter 3 - The space of linear maps & matrices

⑧ **Norms**: $A \in L(\mathbb{R}^n, \mathbb{R}^k)$ $\qquad A = (a_{ij})$

- $\|(a_{ij})\|_F = \left(\sum_{i=1}^{k}\sum_{j=1}^{n} a_{ij}^2\right)^{\frac{1}{2}}$ [Frobenius norm]
- $\|A\| = \sup_{x \in \mathbb{R}^n\setminus\{0\}} \frac{|Ax|}{|x|} = \sup_{|x|=1} |Ax|$ [operator norm]

Properties:
(a) $\|A\| = 0 \iff A = 0$
(b) $\|\alpha A\| = |\alpha|\,\|A\|$
(c) $\|A+B\| \le \|A\| + \|B\|$ $\quad$ Pf: $|(A+B)x| = |Ax + Bx| \le |Ax| + |Bx| \le (\|A\| + \|B\|)|x|$
(d) $\|AB\| \le \|A\|\,\|B\|$

⑨ **Continuity**: Exactly the same as before, just w/ our new norm!

$$x \overset{f}{\longmapsto} \begin{pmatrix} a_{11}(x) & \cdots & a_{1n}(x) \\ \vdots & \ddots & \vdots \\ a_{k1}(x) & \cdots & a_{kn}(x) \end{pmatrix} : U \longrightarrow \mathbb{R}^{k\times n}$$

continuous at x iff $\forall i,j$ $\; x \longmapsto a_{ij}(x)$ cts [componentwise cty $\iff$ cty]

Determinant $\Delta : \mathbb{R}^{n\times n} \longrightarrow \mathbb{R}$, $\Delta(a_{ij}) = \det(a_{ij})$ cts $\because$ polynomial degree n in n^2 variables $a_{11}, \dots, a_{1n}, a_{21}, \dots, a_{2n}, \dots, a_{n1}, \dots, a_{nn}$

⑩ **General linear group**:

easy to check group...

- $GL(n,\mathbb{R}) = \{A \in L(\mathbb{R}^n) : A \text{ is invertible}\} = \{(a_{ij}) \in \mathbb{R}^{n\times n} : \det(a_{ij}) \neq 0\}$

HARD

- $GL(n,\mathbb{R})$ is an open subset of $\mathbb{R}^{n\times n}$

Pf: ① $GL(n,\mathbb{R}) = \Delta^{-1}(\mathbb{R}\setminus\{0\})$ ② Δ cts ③ $\mathbb{R}\setminus\{0\}$ open & $f^{-1}(\text{open}) = \text{open}$

$\{B \in L(\mathbb{R}^n) : \|B - A\| < \alpha\} \subset GL(n,\mathbb{R})$

- Size of an open ball in $GL(n,\mathbb{R})$ [how much wiggle room do we have?]
Given $A \in GL(n,\mathbb{R})$, if $B \in GL(n,\mathbb{R})$ and $\|A - B\| \le \alpha = \frac{1}{\|A^{-1}\|} \Rightarrow B$ invertible
Moreover

$$\|B - A\| < \alpha \Rightarrow \|B^{-1}\| \le \frac{1}{\alpha - \|B - A\|}$$

measure of injectivity

Pf: $x = A^{-1}(Ax) \Rightarrow |x| \le \|A^{-1}\|\,|Ax| \Rightarrow \alpha|x| \le |Ax| \quad \forall x \in \mathbb{R}^n$
If $x \neq 0$ & $\|B - A\| < \alpha = \frac{1}{\|A^{-1}\|}$

$$|Bx| = |Bx - Ax + Ax| \ge |Ax| - |(B-A)x| \ge (\alpha - \|B - A\|)|x| > 0$$

So $Bx \neq 0 \Rightarrow \ker(B) = \{0\} \Rightarrow B \in GL(n,\mathbb{R})$. Replace x by $B^{-1}x$ in above

$$|x| = |B(B^{-1}x)| \ge (\alpha - \|B - A\|)|B^{-1}x|$$

HARD

- $A \longmapsto A^{-1} : GL(n,\mathbb{R}) \longrightarrow GL(n,\mathbb{R})$ is continuous.

Pf: $A^{-1} - B^{-1} = A^{-1}BB^{-1} - A^{-1}AB^{-1} = A^{-1}(B-A)B^{-1}$
$\Rightarrow \|A^{-1} - B^{-1}\| \le \|A^{-1}\|\,\|B - A\|\,\|B^{-1}\|$. Set $\alpha = \frac{1}{\|A^{-1}\|}$. Given $\varepsilon > 0$ set $\delta = \min\{\frac{\alpha}{2}, \varepsilon\}$
Then $\|B - A\| < \delta \Rightarrow \|B^{-1}\| \le \frac{2}{\alpha} \Rightarrow \|A^{-1} - B^{-1}\| \le \frac{2\varepsilon}{\alpha^2}$

uniform cty on U

⑪ **Lipschitz**: $f : U \to \mathbb{R}^k$ Lipschitz cts on U if $\exists M > 0$ s.t. $|f(x) - f(y)| < M|x - y| \quad \forall x, y \in U$
e.g. $A \in L(\mathbb{R}^n)$ $\quad |Ax - Ay| \le \|A\|\,|x - y| \Rightarrow$ linear maps Lipschitz cts.

Chapter 4 - The Derivative

non-linear map $x \mapsto f(x+h)$ is best approximated by the affine linear map $x \mapsto f(x) + Ah$

⑫ Directional Derivative: $\partial_v f(x+tv)\big|_{t=0}$

⑬ The (Fréchet) Derivative: $f: U \to \mathbb{R}^k$ differentiable at $x \in U$ if $\exists A \in L(\mathbb{R}^n, \mathbb{R}^k)$ s.t.

$$\lim_{h\to 0} \frac{|f(x+h) - f(x) - Ah|}{|h|} = 0 \qquad \boxed{A = Df(x)}$$

- Uniqueness:

Pf: ① Suppose $\exists A, B \in L(\mathbb{R}^n, \mathbb{R}^k)$. $\overbrace{\lim_{h\to 0} \frac{|(B-A)h|}{|h|}}^{(*)} \leq \lim_{h\to 0} \frac{|f(x+h) - f(x) - Ah|}{|h|}$ + same 4 $B = 0$

② Set $\Lambda = B - A$. From $(*) = 0$ WTS $\|\Lambda\| = 0 \Rightarrow \Lambda = 0$. (need for operator norm def.)

$(*) \Rightarrow$ Given $\varepsilon > 0$ $\exists \delta$ s.t. $0 < |h| < \delta \Rightarrow \frac{|\Lambda h|}{|h|} < \varepsilon$. Take $y \in S^{n-1}$.

Set $h_y = \frac{\delta}{2} y \Rightarrow |h_y| = \frac{\delta}{2} \Rightarrow \frac{|\Lambda h_y|}{|h_y|} < \varepsilon$. Note: $\Lambda h_y = \frac{\delta}{2}\Lambda y$

③ $|\Lambda y| = \frac{2}{\delta}|\Lambda h_y| < \frac{2}{\delta}\cdot|h_y|\,\varepsilon = \varepsilon \Rightarrow |\Lambda y| < \varepsilon \ \forall y \in S^{n-1} \Rightarrow \|\Lambda\| = 0$

- Differentiability implies continuity.

To calculate from def, consider $f(x+h) - f(x)$ & look for terms linear in h

[Then make guess & prove right from limit]

Pf: ① f diff at $x \Rightarrow \forall \varepsilon > 0\ \exists \delta > 0$ s.t. $|h| < \delta \Rightarrow |f(x+h) - f(x) - Df(x)h| < \varepsilon|h|$

$\cdots \Rightarrow |f(x+h) - f(x)| \leq (\|Df(x)\| + \varepsilon)|h|$

② Set $\delta_* = \min\left\{\frac{\varepsilon}{\|Df(x)\| + \varepsilon}, \delta\right\}$. $|h| < \delta_* \Rightarrow |f(x+h) - f(x)| \leq \cdots < \varepsilon$

⑭ Derivative & directional derivative relation: If $Df(x)$ exists $\Rightarrow \partial_v f(x)$ exists $\forall v \in \mathbb{R}^n$.

$\partial_v f(x) = Df(x)v$ [f differentiable at $x \Rightarrow \partial_v f(x)$ linear in v $\quad \partial_{av+bw} f = a\partial_v f + b\partial_w f$]

[linearity from linearity of $Df(x)$]

Pf: $v \neq 0$ $h \to tv \Rightarrow \lim_{t\to 0} \frac{f(x+tv) - f(x) - Df(x)(tv)}{t|v|} = 0 \Rightarrow \lim_{t \to 0} \frac{f(x+tv) - f(x)}{t} = Df(x)v$

$Df(x)[av + bw] = \cdots$

or chain rule: $\partial_v f(x) = \frac{d}{dt} f(x+tv)\big|_{t=0} = Df(x)\frac{d}{dt}(x+tv)\big|_{t=0} = Df(x)v$ ■ $\Rightarrow \partial_v f(x) = Df(x)v$

⑮ Partial derivatives, gradient, jacobian matrix:

$v_i = \begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix}$ ← 1 in ith position

- $\partial_i f(x) = \{$directional derivative in direction e_i ← basis vector$\}$
- Jacobian matrix at x $Df(x)$ of $f: U \to \mathbb{R}^k$ $\quad [f(x) = (f_1(x), \ldots, f_k(x))]$

$$\partial f(x) = \begin{pmatrix} \partial_1 f_1(x) & \cdots & \partial_n f_1(x) \\ \vdots & \ddots & \vdots \\ \partial_1 f_k(x) & \cdots & \partial_n f_k(x) \end{pmatrix}$$

- Gradient at x, $\nabla f(x)$ for $f: U \to \mathbb{R}$ is $\nabla f(x) = \begin{pmatrix} \partial_1 f(x) \\ \vdots \\ \partial_n f(x) \end{pmatrix} = (\partial f(x))^T$
- If $U \to \mathbb{R}^k$ differentiable at $x \in U$, $h \in \mathbb{R}^n$,

$$\underbrace{Df(x)}_{\text{linear map}}h = \underbrace{\partial f(x)}_{\text{matrix [not standard basis]}} h$$

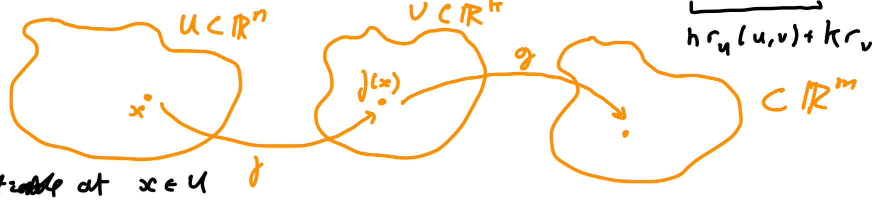
Pf: $h = h_1 v_1 + \ldots + h_n v_n$ so linearity of $Df(x)$:

$$Df(x)h = \sum_{i=1}^n h_i Df(x)v_i = \sum_{i=1}^n h_i \partial_i f(x) = \text{``}\partial_v f(x)\text{''} = \partial f(x) h$$

(16) **Geometric approximation** $\quad U\subset\mathbb{R}^2$

- Tangent to a curve: $r:[a,b]\longrightarrow\mathbb{R}^k$: $\overbrace{r(t+h)\approx r(t)+\partial r(t)\,h}^{\text{tangent line}}$ $\qquad \partial_r=\begin{pmatrix} x_u & x_v\\ y_u & y_v\\ z_u & z_v\end{pmatrix}$
- Tangent (plane) to a surface: $r: U\longrightarrow\mathbb{R}^k$: $r(u+h,v+k)=r(u,v)+\underbrace{\partial r(t)(h,k)}_{h r_u(u,v)+k r_v(u,v)}$

(17) **Chain rule**



Let $U\subset\mathbb{R}^n$ open
$V\subset\mathbb{R}^k$ open,
$f: U\rightarrow\mathbb{R}^k$ differentiable at $x\in U$
$f(x)\in V$ and $g: V\longrightarrow\mathbb{R}^m$ differentiable at $f(x)$ $\Rightarrow$ $g\circ f:\mathbb{R}^n\longrightarrow\mathbb{R}^m$ differentiable at x

$$\underbrace{D(g\circ f)(x)=Dg(f(x))Df(x)}_{\text{linear maps}}\quad\text{or}\quad\underbrace{\partial g\circ f(x)=\partial g(f(x))\cdot\partial f(x)}_{\text{matrices}}$$

can use in PDEs to verify solutions...

Common special cases:

- $f:\mathbb{R}^n\longrightarrow\mathbb{R}$, $g:\mathbb{R}\longrightarrow\mathbb{R}$, $\nabla(g(f(x)))=g'(f(x))\,\nabla f(x)$
- $\nabla|x|=\frac{x}{|x|}$ for $x\in\mathbb{R}^n$. [set $f(x)=|x|^2$, $g(t)=\sqrt{t}$, $g\circ f$]
- $r:\mathbb{R}\longrightarrow\mathbb{R}^n$, $f:\mathbb{R}^n\longrightarrow\mathbb{R}$ $\quad \nabla f(r(t))=\nabla f(r(t))\cdot r'(t)$

Lemmas needed in proof:

- $f: U\longrightarrow\mathbb{R}^k$, $x\in U$, $r>0$ s.t. $B(x,r)\subset U$. $A\in L(\mathbb{R}^n,\mathbb{R}^k)$. Define $\Delta_{x,A}f: B(0,r)\longrightarrow\mathbb{R}^k$ by

$$\Delta_{x,A}f(h)=\begin{cases}\frac{f(x+h)-f(x)-Ah}{|h|} & \text{if } h\neq 0\\ 0 & \text{if } h=0\end{cases}$$

Similar to local linearisation lemma from analysis II

Then f is differentiable at x with $Df(x)=A$ $\Leftrightarrow$ $\Delta_{x,A}f$ cts at 0.

Pf: $\Delta_{x,A}f(h)$ cts at 0 $\Leftrightarrow$ $\lim_{h\to 0}\Delta_{x,A}f(h)=\Delta_{x,A}f(0)=0$ $\Leftrightarrow$ $Df(x)$ exists and equals A.

- Let $\tau>0$, $\delta: B_\tau\subset\mathbb{R}^n\longrightarrow\mathbb{R}^k$ w/

$$\delta(h)=\begin{cases}\xi(h)\eta(h) & 0<|h|<\tau\\ 0 & \delta=0\end{cases}$$

$\xi: B_\tau\setminus\{0\}\longrightarrow\mathbb{R}$ bdd, $\eta: B_\tau\longrightarrow\mathbb{R}^k$ cts at $0\in B_\tau$, $\eta(0)=0$ $\Rightarrow$ δ cts at $0\in B_\tau$

Pf: ① η cts at 0 $\Rightarrow$ given $\varepsilon>0$ $\exists\sigma\in(0,\tau)$ s.t. $|h|<\sigma$ $\Rightarrow$ $|\eta(h)|<\varepsilon$
② ξ bdd $\Rightarrow$ $\exists M>0$ s.t. $|\xi(h)|<M$ $\forall h\in B_\tau\setminus\{0\}$
③ so $0<|h|<\sigma$ $\Rightarrow$ $|\delta(h)|<M\varepsilon$, that is $\lim_{h\to 0}\delta(h)=0=\delta(0)$ $\Leftrightarrow$ δ cts at 0.

Pf: ① Go in lines along axis ② MVT ③ continuity

(18) **Continuity of partial derivatives $\Rightarrow$ differentiability**: $f: U\longrightarrow\mathbb{R}^k$.

∂f exists on whole ball & f cts at x $\Downarrow$ f diff at x.

Suppose jacobian matrix $\partial f(y)$ exists $\forall y\in B(x,r)\subset U$ and f cts at x.
Then f diff at x & $Df(x)h=\partial f(x)\,h$ (suppose this exists) $\forall h\in\mathbb{R}^n$

Pf: $(n=2,k=1)$ Define $\Delta f(h_1,h_2)=f(x_1+h_1,x_2+h_2)-f(x_1,x_2)-\overbrace{(h_1\partial_1 f(x_1,x_2)+h_2\partial_2 f(x_1,x_2))}^{\text{our guess for } Df(x)}$ $\quad$ WTS $\lim_{h\to 0}\frac{\Delta f(h)}{|h|}=0$

① Only know PDs along axis. Split! $f(x_1+h_1,x_2+h_2)-f(x_1,x_2)=\underbrace{[f(x_1+h_1,x_2+h_2)-f(x_1+h_1,x_2)]}_{\mathrm{I}}+\overbrace{[f(x_1+h_1,x_2)-f(x_1,x_2)]}^{\mathrm{II}}$
[x_2 fixed]

② MVT on $f(\cdot,x_2)$: $\exists\theta_1\in(0,1)$ s.t. $\mathrm{II}=f(x_1+h_1,x_2)-f(x_1,x_2)=h_1\partial_1 f(x_1+\theta_1 h_1,x_2)$
MVT on $f(x_1+h_1,\cdot)$: $\exists\theta_2\in(0,1)$ s.t. $\mathrm{I}=f(x_1+h_1,x_2+h_2)-f(x_1+h_1,x_2)=h_2\partial_2 f(x_1+h_1,x_2+\theta_2 h_2)$

③ Substitute: $\Delta f(h_1,h_2)=h_1[\partial_1 f(x_1+\theta_1 h_1,x_2)-\partial_1 f(x_1,x_2)]+h_2[\partial_2 f(x_1+h_1,x_2+\theta_2 h_2)-\partial_2 f(x_1,x_2)]$

④ continuity $\partial_1 f$ and $\partial_2 f$ at (x_1,x_2): Given $\varepsilon>0$ $\exists\delta>0$ ($<r$) s.t. $|(\tilde h_1,\tilde h_2)|<\delta$ $\Rightarrow$ cty about P.D. (2)
$|(h_1,\theta_2 h_2)|<|(h_1,h_2)|$ & $|(\theta_1 h_1,h_2)|<|(h_1,h_2)|$ so by cty

$$|\Delta f(h_1,h_2)|<(|h_1|+|h_2|)\cdot\varepsilon\leq\varepsilon\sqrt{2}\,|(h_1,h_2)|\;\Rightarrow\;\lim_{h\to 0}\frac{\Delta f(h)}{|h|}=0\;\Rightarrow\; Df(x)\text{ exists!}$$

■

[C']

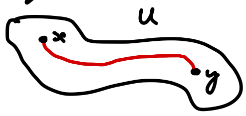
19. Continuously differentiable functions:

- **Def:** Suppose $f: U \to \mathbb{R}^k$ diff on U. f ctsly diff at p if $Df(x)$ cts at p.
- $f: U \to \mathbb{R}^k$ ctsly diff on $U \iff \partial f: U \to \mathbb{R}^{k\times n}$ cts on U

  $\partial_i f_j$ all cts

  Pf: ...
- Can practically check cts differentiability by computing PDs & checking cts.

20. Mean Value Inequality+: $x, y \in U$ joined by cts diff path $r:[a,b] \to U$ $r(a)=x$, $r(b)=y$. $f \in C'(U, \mathbb{R}^k)$ & $\exists M>0$ s.t. $\|\partial f(x)\| < M$ $\forall x \in U$. Then



$$|f(x)-f(y)| \leq M \cdot \text{length}(C_{xy}) \qquad C_{xy} = r([a,b])$$

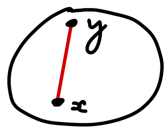
Pf: $|f(y)-f(x)| = |f(r(b)) - f(r(a))|$

$$\overset{(FTC)}{=} \left|\int_a^b \frac{d}{dt} f(r(t))\,dt\right| \overset{\text{[chain rule]}}{=} \left|\int_a^b \partial f(r(x))\, r'(t)\,dt\right| \leq \int_a^b \|\partial f(r(x))\|\,|r'(t)|\,dt \leq M \underbrace{\int_a^b |r'(t)|\,dt}_{\text{length of } C_{xy}}$$

Corollaries:

- U differentiably path connected & $\partial f(x) = 0$ $\forall x \in U \Rightarrow f$ constant on U.

  Pf: Fix $y \in U$. given $x \in U$ $\exists$ path between. $M=0$ in proof above.



- $U \subset \mathbb{R}^n$ convex if $\forall x, y \in U$, line $L_{xy} \subset U$
- $U \subset \mathbb{R}^n$ convex, $\|\partial f(x)\| < M \Rightarrow |f(x) - f(y)| < M|x-y|$

  Pf: $\text{length}(L_{xy}) = |x-y|$

21. (Derivative bound) $\underset{\not\Leftarrow}{\Rightarrow}$ (Lipschitz), $\partial_i f = 0 \not\Rightarrow f$ doesn't depend on x_i

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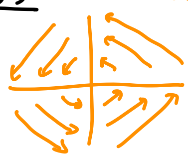
Chapter 5 - Vector Fields, line/surface integrals

$U \subset \mathbb{R}^n$ open, path connected



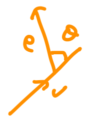
underline $\because$ $\underline{v}(x)$ is a vector "attatched" at x.

22. Definitions:

- Vector field $\underline{v}: U \to \mathbb{R}^n$ $\quad \underline{v}(x) = \begin{pmatrix} v_1(x) \\ \vdots \\ v_n(x) \end{pmatrix}$

  curve can have multiple parametrisations.
- path is $r:[a,b] \to \mathbb{R}^n$. Curve is image of a path w/ specified endpoints. C_{pq}
- path $r:[a,b] \to \mathbb{R}^n$ is regular if $r'(t) \neq 0$ $\forall t \in [a,b]$. Im(regular path) = regular curve.
- If $r:[a,b] \to \mathbb{R}^n$ is a regular param of a curve $C \subset \mathbb{R}^n$,

  $$L := \text{length}(C) = \int_a^b |r'(t)|\,dt$$



$v \cdot e = |v|\cos\theta$

- Component$_u$ of $v \in \mathbb{R}^n$ in direction of unit vector $e \in \mathbb{R}^n$ is $v \cdot e$.
- $\rho(u) = \int_a^u |r'(t)|\,dt$ is arc length parametrization. $\rho: [0, L] \to \mathbb{R}^n$.

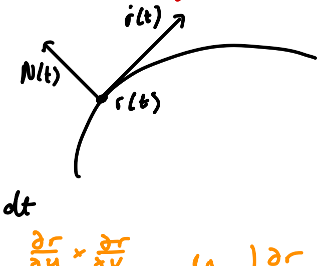
  Unit tangent to C_{pq} at $\rho(s)$ is $\dot\rho(s) = \frac{\partial \rho}{\partial s}(s)$. $\underline{v}(\rho(s)) \cdot \dot\rho(s)$ is tangential component of $\underline{v}$ on C_{pq}
- **Tangential line integral:** of $\underline{v}$ along C_{pq}

  $$\underbrace{\int_0^L \underline{v}(\rho(s)) \cdot \dot\rho(s)\,ds}_{\text{definition}} = \underbrace{\int_a^b \underline{v}(r(t)) \cdot \frac{dr}{dt}\,dt}_{\text{compute}}$$

  $$\boxed{\int_C f\,ds = \int_a^b f(r(u))\,|r'(u)|\,du}$$

  line integral of f along C
- Orientation

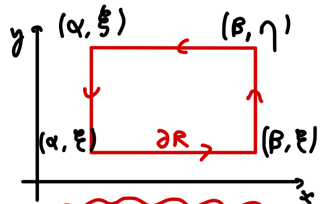
  $$\int_{C_{pq}} \underline{v} \cdot dr = -\int_{C_{qp}} \underline{v} \cdot dr$$



- **In the plane:**
  - $r(t) = (x(t), y(t))$, $\dot r(t) = \left(\frac{dx}{dt}, \frac{dy}{dt}\right)$, $N(t) = \dot r(t)^\perp = \left(\frac{dy}{dt}, -\frac{dx}{dt}\right)$
  - Flux of $\underline{v}$ across a curve C $\int_0^L v(\rho(s)) \cdot n(s)\,ds = \int_a^b \underline{v}(r(t)) \cdot N(t)\,dt$
- **Surface in $\mathbb{R}^3$:**
  - $N(u,v) = \frac{\partial r}{\partial u} \times \frac{\partial r}{\partial v}$ where $r(u,v)$ is surface param.

    $n(u,v) = \dfrac{\frac{\partial r}{\partial u} \times \frac{\partial r}{\partial v}}{\left|\frac{\partial r}{\partial u} \times \frac{\partial r}{\partial v}\right|}$, $dA = \left|\frac{\partial r}{\partial u} \times \frac{\partial r}{\partial v}\right| du\,dv$
  - $\underline{v}$ vector field, Surface S

$$\iint_S \underline{v} \cdot n\,dS = \text{Flux } \underline{v} \text{ across } S := \iint_S \underline{v} \cdot n\,dA = \iint_U v(r(u,v)) \cdot \left(\frac{\partial r}{\partial u} \times \frac{\partial r}{\partial v}\right) du\,dv$$



Chapter 6 - The Integral Theorems of vector calculus

(23) **Green's thm for a rectangle:** $\underline{v}$ planar vector field $(a(x,y), b(x,y))$

$$\int_{\partial R} \underline{v}\cdot dr = \underbrace{\int_\alpha^\beta a(x,\xi)dx - \int_\alpha^\beta a(x,\eta)dx}_{FTC} + \underbrace{\int_\xi^\eta b(\beta,y)dy - \int_\xi^\eta b(\alpha,y)dy}_{FTC}$$

$$= \int_\alpha^\beta \int_\xi^\eta -\frac{\partial a}{\partial y}\, dy\, dx + \int_\xi^\eta \int_\alpha^\beta \frac{\partial b}{\partial x}\, dx\, dy = \iint_R \left\{\frac{\partial b}{\partial x} - \frac{\partial a}{\partial y}\right\} dx\, dy$$

To extend, need to make precise idea of ① regions ② normal

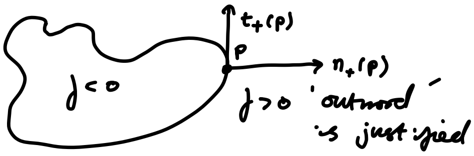
(24) **Regions & Unit normal:**

defining function of set Ω

- A region in $\mathbb{R}^n$ is a bdd open subset $\Omega \subset \mathbb{R}^n$ s.t. $\exists f: \mathbb{R}^n \longrightarrow \mathbb{R}$ w/ following
  - ⓐ all partial derivatives of f cts
  - ⓑ $\Omega = \{x \in \mathbb{R}^n : f(x) < 0\}$
  - ⓒ $\nabla f(p) \neq 0 \quad \forall p \in f^{-1}\{0\} = \{x \in \mathbb{R}^n : f(x) = 0\}$

outward unit normal to Ω at p.

- $n_+(p) := \dfrac{\nabla f(p)}{|\nabla f(p)|}$ [(c) lets us define this]



Note: $\exists \delta > 0$ s.t. $f(p + t n_+(p)) > 0$ for $0 < t < \delta$, $f(p + t n_+(p)) < 0$ for $-\delta < t < 0$

- $\partial\Omega = f^{-1}\{0\}$ is the boundary of Ω. $[\Omega \cup \partial\Omega = \overline{\Omega}]$

(25) **In the plane:**

- **Green's thm in the plane:**
  - $\underline{v} = (a, b)$, $\quad \mathrm{curl}(\underline{v}) = \dfrac{\partial b}{\partial x} - \dfrac{\partial a}{\partial y}$
  - $\Omega \subset \mathbb{R}^2$ region in $\mathbb{R}^2$. Regular parametrisation $r: [a,b] \longrightarrow \mathbb{R}^2$ of $\partial\Omega$ is positively oriented if $t_+ := \dfrac{r'}{|r'|}$ is a positively oriented unit tangent vector to $\partial\Omega$ at $r(t)$. $t_+ = -(n_+(p))^\perp$

e.g. "statement of Green's thm, carefully defining all terms involved"

  - (Green's thm for planar regions): $\Omega \subset \mathbb{R}^2$ region. $\underline{v}: U \longrightarrow \mathbb{R}^2$ be a ctsly diff planar vector field on U which contains $\overline{\Omega} = \Omega \cup \partial\Omega$. Then circulation

$$\iint_\Omega \mathrm{curl}((v(x,y))\, dA_{x,y} = \oint_{\partial\Omega} \underline{v}\cdot t_+\, ds = \oint_{\partial\Omega} \underline{v}\cdot dr$$

where s is the arc length parameter along $\partial\Omega$, r is a positively oriented parametrisation of $\partial\Omega$ and the area $dA_{x,y}$ can be written as $dx\,dy$.

- **Flux & Divergence in the plane:**
  - $\mathrm{div}(\underline{v}) = \nabla\cdot\underline{v} = \dfrac{\partial v_1}{\partial x_1} + \dots + \dfrac{\partial v_n}{\partial x_n}$
  - (Gauss' thm / Divergence thm for planar regions): Ω a region in $\mathbb{R}^2$. $\underline{v}: U \longrightarrow \mathbb{R}^2$ ctsly diff planar vector field on $U \supset \overline{\Omega}$. Then

$$\iint_\Omega \nabla\cdot\underline{v}\, dA = \int_{\partial\Omega} \underline{v}\cdot n_+\, ds$$

Pf: apply Green's thm to $\underline{v}^\perp$

where n_+ is the unit outward normal to Ω.

$$\psi(r,\theta,z) = \begin{pmatrix} r\cos\theta \\ r\sin\theta \\ z \end{pmatrix} \quad \begin{matrix} r \in [1,2] \\ \theta \in [0, 2\pi] \\ z \in [0, 2-r] \end{matrix}$$

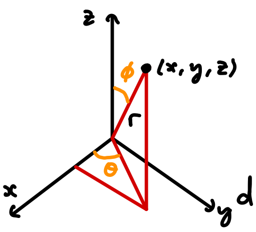
z coordinate can be r dependent

(26) **Flux & Divergence in $\mathbb{R}^3$:**

- (Divergence thm in $\mathbb{R}^3$): Ω a region in $\mathbb{R}^3$. $\underline{v}: U \longrightarrow \mathbb{R}^3$ C^1 vector field on $U \supset \overline{\Omega}$. Then

$$\iiint_\Omega \nabla\cdot\underline{v}\, dV = \iint_{\partial\Omega} \underline{v}\cdot n_+\, dA$$

where n_+ is the outward pointing unit normal to Ω, dV is volume element of Ω.



spherical	cylindrical
$x = r\cos\theta\sin\phi$ $y = r\sin\theta\sin\phi$ $z = r\cos\phi$	$x = r\cos\theta$ $y = r\sin\theta$ $z = z$
$dV = r^2\sin\phi\, dr\, d\theta\, d\phi$	$dV = r\, dr\, d\theta\, dz$

If you parametrise a volume by $\psi(r,\theta,z)$, then

$$dV = |\det D\psi(r,\theta,z)|\, dr\, d\theta\, dz$$

absolute value of the determinant of the jacobian matrix for ψ.

(27) **Vector field concepts:**

- if a vector field $\underline{v} = \nabla f$, $f: U \longrightarrow \mathbb{R}$. $\underline{v}$ is a gradient field. f is the scalar potential.
- (FTC for gradient field): $f: U \longrightarrow \mathbb{R}$ its diff, $C_{pq} \subset U$

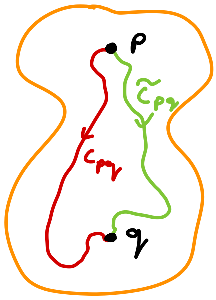
$$\int_{C_{pq}} \nabla f \cdot dr = f(q) - f(p) = f(r(b)) - f(r(a)) = \int_a^b \frac{d}{dt} f(r(t))\,dt = \int_a^b \nabla f(r(t)) \cdot r'(t)\,dt$$

- If C is a closed curve $\oint_C \nabla f \cdot dr = 0$

Conservative vector fields

- $\underline{v}$ is conservative if $\oint_C \underline{v} \cdot dr = 0$ $\forall$ closed curves C
- $\underline{v}$ is conservative $\Longleftrightarrow$ $\forall p, q \in U$

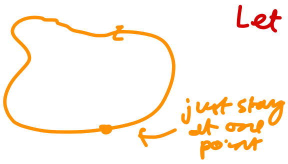
$$\int_{C_{pq}} \underline{v} \cdot dr \text{ is independent of choice of curve } C_{pq} \text{ in } U$$



Pf: '$\Longrightarrow$' Let C_{pq} and $\tilde{C}_{pq}$ two curves connecting p and q.
construct a closed curve $\underset{\gamma}{C}$: $p \xrightarrow[r]{C_{pq}} q \xrightarrow[-\tilde{C}_{pq}]{\tilde{r}} p$

$$\underline{v} \text{ conservative} \Rightarrow 0 = \oint_C \underline{v} \cdot d\gamma = \int_{C_{pq}} \underline{v} \cdot dr - \int_{\tilde{C}_{pq}} \underline{v} \cdot d\tilde{r}$$

'$\Longleftarrow$' Take C closed curve. param: $\gamma : [0,1] \longrightarrow U$ $\gamma(0) = \gamma(1)$.
Let $r(t) = \gamma(0)$ $\forall t$ $\qquad$ $\gamma(t)$ & $r(t)$ have same start & end pts.



$$\oint_C \underline{v} \cdot dr = \int_0^1 \underline{v}(\gamma(t)) \cdot \frac{d\gamma}{dt}\,dt = \int_0^1 \underline{v}(\gamma(t)) \cdot \cancel{\frac{dr}{dt}} = 0 \quad \left[\text{as } \frac{dr}{dt} = 0\right]$$

- $\underline{v} : U \longrightarrow \mathbb{R}^n$ is a gradient field $\Longleftrightarrow$ $\underline{v}$ is conservative. $\left[v = \nabla f \Leftrightarrow \oint_C v \cdot dr = 0 \right]$

Pf: '$\Longrightarrow$' $v = \nabla f \Rightarrow \oint_C \nabla f \cdot dr = 0$ as C closed.

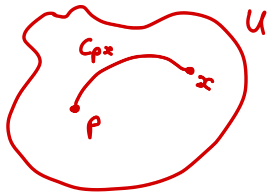
'$\Longleftarrow$' If f conservative, pick $p \in U$. Define $f : U \to \mathbb{R}$ by

$\boxed{\text{WTS } \nabla f = \underline{v}}$

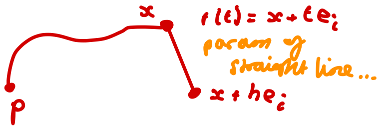
$\Updownarrow$

$\boxed{\frac{\partial f}{\partial x_i} = v_i \quad i = 1, \dots, n}$

$$f(x) = \int_{C_{px}} \underline{v} \cdot dr \qquad \left[\text{doesn't depend on path} \Rightarrow \text{unambiguously defined}\right]$$



$$f(x + te_i) = \underbrace{\int_{C_{px}} \underline{v} \cdot dr}_{f(x)} + \underbrace{\int_0^h \underline{v}(r(t)) \cdot \frac{dr}{dt}\,dt}_{\int_0^h v(r(t)) \cdot e_i\,dt}$$



div$(v) = 0$
$\Updownarrow$
v "divergence free"

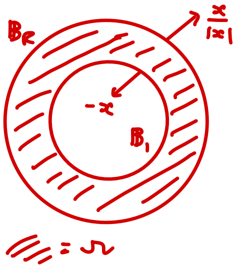
$$\text{so } \frac{\partial f}{\partial x_i} = \lim_{h \to 0} \frac{f(x + te_i) - f(x)}{h} = \lim_{h \to 0} \left(\frac{1}{h} \int_0^h v_i(r(t))\,dt \right) = \frac{d}{dh} \left| \underbrace{\int_0^h v_i(r(t))\,dt}_{v_i(r(0))} \right|_{h=0} = v_i(x)$$

(28) **Laplacian functions**: $\underset{\text{the laplacian}}{\Delta f} = \nabla \cdot (\nabla f) = \sum_{i=1}^n \frac{\partial^2 f}{\partial x_i^2}$.

- $\Delta f = 0$ is laplace's equation, solutions are harmonic functions.
- $f : \mathbb{R}^n \setminus \{0\} \longrightarrow \mathbb{R}$ is radial if $\exists \varphi : \mathbb{R}_{>0} \longrightarrow \mathbb{R}$ s.t. $f(x) = \varphi(|x|)$ $\forall x \in \mathbb{R}^n \setminus \{0\}$

$$\nabla f(x) = \varphi'(|x|) \nabla |x| = \varphi'(|x|) \frac{x}{|x|}$$

- Gradient of harmonic functions divergence free. $\nabla \cdot \underline{v} = \nabla \cdot (\nabla f) = \Delta f = 0$



$$0 = \iiint_\Omega \nabla \cdot \underline{v}\,dV = \iint_{\partial\Omega} \underline{v} \cdot n_+\,dA$$

$$0 = \iint_{\partial\Omega} \underline{v} \cdot n_+\,dA = \iint_{\partial B_1 \cup \partial B_R} \nabla f(x) \cdot n_+\,dA$$

$$= \iint_{\partial B_R} \varphi'(R) \frac{x}{R} \cdot \frac{x}{R}\,dA + \iint_{\partial B_1} \varphi'(1) \frac{x}{R} \cdot (-x)\,dA$$

$$\Rightarrow 0 = \varphi'(R) \cdot 4\pi R^2 - \varphi'(1)(4\pi)$$

$$\text{ODE: } \varphi(r) = -\frac{\varphi'(1)}{r} + b \Rightarrow f(x) = \frac{a}{|x|} + b$$

radial harmonic functions on $\mathbb{R}^3$ have this form.

Chapter 7 - Second Order Derivatives

Note: classification non-examinable

- $L(\mathbb{R}^n, \mathbb{R}) = (\mathbb{R}^n)^*$ [space of linear functionals on $\mathbb{R}^n$]

(29) The Hessian:

- $f: U \to \mathbb{R}$ diff at x, Df diff at $x \in U \Rightarrow \exists H_x \in L(\mathbb{R}^n, (\mathbb{R}^n)^*)$ s.t.

$$\lim_{h \to 0} \frac{\| Df(x+h) - Df(x) - H_x h \|}{|h|} = 0 \qquad \boxed{H_x = D^2 f}$$

$$\text{Hess} f(x) = \partial^2 f(x) = \begin{pmatrix} \partial_{11} f(x) & \cdots & \partial_{1n} f(x) \\ \vdots & \ddots & \vdots \\ \partial_{n1} f(x) & \cdots & \partial_{nn} f(x) \end{pmatrix} = \begin{pmatrix} \frac{\partial^2 f}{\partial x_1^2}(x) & \cdots & \frac{\partial^2 f}{\partial x_1 \partial x_n}(x) \\ \vdots & \ddots & \vdots \\ \frac{\partial^2 f}{\partial x_n \partial x_1}(x) & \cdots & \frac{\partial^2 f}{\partial x_n^2}(x) \end{pmatrix}$$

- $D^2 f(x)$ exists $\Rightarrow$ 2nd order partial derivatives commute.
- $C^k(U, \mathbb{R}) = \{ f: U \to \mathbb{R}^m :$ all derivatives of f upto & inc k exist & cts at $x \}$

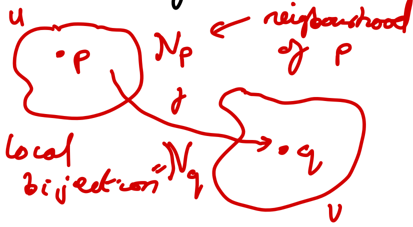
Chapter 8 - Inverse Function Theorem

$\psi: U \to V$
Diffeomorphism: bijection, diff on U, inverse cts diff on V

(30) Derivatives of inverses: $\psi: U \to V$ bijection. Diff at $x \in U$. ψ^{-1} diff at $y = \phi(x) \in V$.
Then $D\psi(x)$ and $D\psi^{-1}(y)$ invertible & ...

$$\psi(\psi^{-1}(y)) = y \Rightarrow D\psi(\psi^{-1}(y)) \circ D\psi^{-1}(y) = I_n \Rightarrow (D\psi^{-1})(y) = (D\psi(\psi^{-1}(y)))^{-1}$$

(31) Inverse Function Thm: $U \subset \mathbb{R}^n$ open, $\psi \in C^1(U, \mathbb{R}^n)$. Need to assume that $D\psi(p)$ invertible at $p \in U$ [$\det \partial \psi(p) \neq 0$]
Set $\psi(p) = q$.



(i) Then have neighbourhoods around p & q s.t. $\psi: N_p \to N_q$ locally bijective.

(ii) $\psi^{-1}: N_q \to N_p$ cts diff and $(D\psi^{-1})(y) = (D\psi(\psi^{-1}(y)))^{-1} \quad \forall y \in N_q$

Note: If the derivative is injective (& cts) $\Rightarrow f$ locally injective

Pf: prop 8.6.1, see Q2 Assignment 3.

$F(x,y) = c$ & solving this implicit equation. what conditions do you need?

Chapter 9 - Implicit Function Theorem

l equations, $n+l$ unknowns

(32) Implicit Function Thm: $\Lambda = (\overset{n}{A} \quad \overset{l}{B}) \} l \qquad z = (\overset{\mathbb{R}^n}{x}, \overset{\mathbb{R}^l}{y}) \qquad F: \mathbb{R}^{n+l} \to \mathbb{R}^l$

$F(x,y) = (A \ B)\begin{pmatrix} x \\ y \end{pmatrix} = Ax + By = c \in \mathbb{R}^l \Rightarrow y = B^{-1}(c - Ax)$ [$\because B$ invertible]

- If we have one solution (x_0, y_0) to $F(x,y) = c$. If $\partial_y F(x_0, y_0) \in \mathbb{R}^{l \times l}$ invertible, then can solve for y in terms of x near x_0. Precisely, $\exists$ open set $N_{x_0} \subset U$, & $\exists g \in C^1(N_{x_0}, \mathbb{R}^l)$ s.t.

why is tangent space $T_z \Gamma_c =$ $= \ker(\partial F(z))$ shifted by $z \in \Gamma_c$?

(i) $g(x_0) = y_0$ & $F(x, g(x)) = c \quad \forall x \in N_{x_0}$

(ii) $\partial_y F(x, g(x))$ invertible, $\partial g(x) = -(\partial_y F(x, g(x)))^{-1} \cdot \partial_x F(x, g(x))$

Pf: $F(x, g(x)) = F(x, y(x,c)) = c \Rightarrow \partial_x F(x, g(x)) + \partial_y F(x, g(x)) \underset{\text{diff wrt } x}{\partial g(x)} = 0$

(33) Level set of a function is a submanifold of Euclidean space when?

Hard... Do examples...

Revision Lecture Checklist

- [ ] Show that $f(x,y) = \frac{x^3}{x^2+y^2}$ if $(x,y) \neq (0,0)$, $f(0,0) = 0$ is cts.
  [as in 4.3.8]

- [ ] Know all these proofs

$$
\begin{tikzcd}
\text{continuity} & & \text{differentiability} \arrow[ll, Rightarrow, "\square"'] \arrow[d, Rightarrow, "\square"] \\
\text{continuity along lines} \arrow[d, Rightarrow, "\square"] \arrow[from=u, Rightarrow, "\square"] & & \text{directional derivative} \arrow[ll, Rightarrow, "\square"'] \arrow[d, Rightarrow, "\square"] \\
\text{separate continuity} & & \text{partial derivatives} \arrow[ll, Rightarrow, "\square"']
\end{tikzcd}
$$

- [ ] Question 3 Example sheet 4 [for T w/ examples]

- [ ] Question 3b Assignment 1] cty in terms of open & closed sets.
- [ ] Question 5 Example sheet 2]

- [ ] Proof of the extreme value theorem] apply Bolzano Weierstrass
- [ ] Question 5 Example sheet 3 [show A is injective]]

- [ ] Review all derivative definitions [differentiability, directional derivative, partial derivatives, gradient, Jacobian matrix]

- [ ] Example 4.5.2. $f(A) = A^2$ what is $Df(A)H$, calculate ker $Df(I_n)$

- [ ] Apply inverse function thm to find square root $p_1 = I_n$, $q = f(I_n) = I_n$
  [see exam review document]

- [ ] 4.6.2, 4.6.3, 4.6.4 [computations using chain rule]

- [ ] Q1, Q2, Q3 Example sheet 5

- [ ] Q1 Assignment 3

- [ ] Proof of generalized mean value inequality [prop 4.8.1]

- [ ] Gradient of a radial function. [formula 6.14]

- [ ] Q1, Q3 Assignment 4

- [ ] Q1 Examples sheet 8

- [ ] Q1 and Q2 Example sheet 7

- ☐ Q2 and Q4 Assignment 4
- ☐ Q2 Example sheet 8
- ☐ Q3 Example sheet 7
- ☐ Q6 Example sheet 9
- ☐ PDE integration by parts Q7 Example sheet 9
- ☐ Q1-5 Example sheet 9 [gradient fields etc]
- ☐ Calculations w/ inverse function theorem. Example 8.5.2
- ☐ Q9, Q10 Example sheet 9.
- ☐ Example 8.6 in notes.
- ☐ Calculations w/ implicit function theorem. Example 9.3.1
- ☐ Local parametrisation of a level set Γ_c by means of implicit function theorem.
- ☐ Relation of $T_z \Gamma_c$ with $\ker \partial F(z)$, $z \in \Gamma_c = F^{-1}\{c\}$
- ☐ Proof of at least one standard theorem.