MA241 Combinatorics 4/10/22

You need to practice by yourself. Lots of extra problems. do them! very important.

Lecture 1

What is Combinatorics?

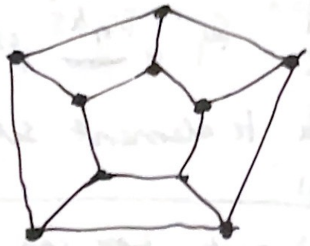
"Counting things." X The study of "finite structures".

Its a problem based field. counting problems & not. $\boxed{\text{Finite Objects.}}$

Combinatorics ⟷ Probability. How many 6 digit lottery numbers are there? If I pick one at random, what is $\mathbb{P}(\text{win})$? Not probability distributions.

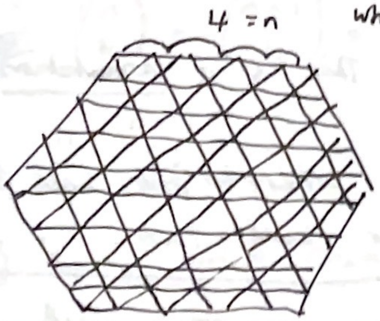
Q I have k identical balls, which I want to distribute in n labeled boxes. How many ways? n^k

Q



Can I walk on the graph visiting each vertex exactly once, ending where I started?

Q



what happens as $n \to \infty$?

How many ways are there to fill up the gameboard with dominos?

Why? It naturally occurs in maths...

connections to:
- linear algebra ?!
- complex analysis ?!
- group theory
- algebraic geometry
- statistical physics, quantum...

] no matrices or complex numbers

Simplified model for crystal formation!

Simple answer...

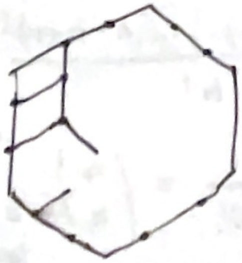
Thm: [MacMahon 1916]. Number is

$$\prod_{i=1}^{n}\prod_{j=1}^{n}\prod_{k=1}^{n}\frac{i+j+k-1}{i+j+k-2}$$

← lots of fractions but it ends up being an integer...

$n=3$: $\frac{2}{1}\cdot\left(\frac{3}{2}\right)^3\left(\cdot\right)\ldots = 980$

You can translate combinatorical questions to make them easier!



Lecture 2

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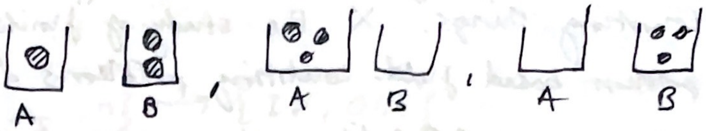
Enumerative Combinatorics

① **Eg** How many ways to distribute k labelled balls in n labelled boxes.

Solve: $k=3,\ n=2$



8 ways to do it.

- all in A
- all in B
- 1 in A, 2 in B
- 1 in B, 2 in A

too complex!

For each ball, either in A or B. Two choices for ball 1, 2 choices for ball 2, two choices for ball 3. Completely independent. So n^k choices.

Same as how many functions from a k element set to an n element set?

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②. **Eg** Same question with $k=n$, where each box can only hold one ball?

$n=3$,

, $(2,3,1)$, $(1,2,3)$. This is a permutation

$\boxed{n!}$ Reasoning: There are n choices for where to put ball one
There are $n-1$ choices for ball two . . . .

---

③ **Eg**. Given 10 people, how many 3 person committees could you form? $n\}$

- 10 choices for ~~committee #1~~ person 1
- 9 choices for person 2
- 8 choices for person 3.

But counted committees more than once! →

CBE
CEB
BCE
CBE
EBC
ECB

← 3! overcount

In general:

- n choices for person 1
- $n-1$ " " " 2
- . . .
- $(n-k+1)$ choices for person k.

But we've counted each permutation $k!$ over.

$$\frac{n(n-1)\cdots(n-k+1)}{k!}$$

$$= \frac{n!}{k!(n-k)!} = \binom{n}{k} \quad {}^nC_k$$

binomial coefficient.

Q: How many committees, 'no size restriction', from n people?

← sum through all possible committees?

$$\boxed{\sum_{k=0}^{n} \binom{n}{k} = 2^n}$$

- **Aside:** subsets of $\{1, 2, \dots, n\}$.

$\updownarrow$

Functions from $\{1, \dots, n\} \to \{IN, OUT\}$

] There are 2^n such possibilities.

Used two different ways of counting to argue two things are the same. Two ways of expressing the same problem.

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1

	$k=0$	1	2	3	4	5
$n=0$						
$n=1$	1	1				
2	1	2	1			
3	1	3	3	1		
4	1	4	6	4	1	
5	1	5	10	10	5	1

Prop: $\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$

Pf: ① $\frac{(n-1)!}{k!(n-k-1)!} + \frac{(n-1)!}{(k-1)!((n-1)-(k-1))!} \overset{\text{algebra}}{=} \frac{n!}{k!(n-k)!}$ ← not combinatorics! algebra doesn't give us understanding for why true! it works;

② same counting problem in two ways.

LHS = k element subsets of an n element set $\{1, 2, \dots, n\}$.

RHS = ?

- First count k element subsets containing 1
- Then count k element subsets not containing 1.

] division is well defined.

↓

count k element subsets of $\{2, 3, \dots, n\}$, this is $\binom{n-1}{k}$

→ we know 1 is in there so pick $k-1$ elements of $\{2, 3, \dots, n\}$

$k-1$ subsets of $\{2, 3, \dots, n\}$ so $\binom{n-1}{k-1}$

Thus $\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$

Binomial Theorem

Let $n \in \mathbb{N}$. $(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$

why: $(a+b)(a+b)\dots(a+b)$

How many ways to get $a^k b^{n-k}$. To choose k as, out of n pick k. out of the n bs pick $n-k$.

Generalisation

Given n people, divide them into groups of sizes $k_1, k_2, \dots, k_r$. where $k_1 + k_2 + \dots + k_r = n$.

How many choices for 1st person on first committee?

$$\boxed{\frac{n!}{k_1!\,k_2!\dots k_r!}} = \binom{n}{k_1, k_2, \dots, k_r}$$ multinomial coefficient.

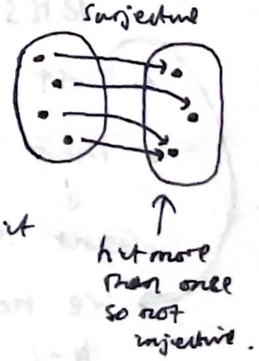
Pick 3 for your group & 7 for the other group with binomial.

$$\frac{n!}{\underset{\substack{\uparrow \\ k_1}}{k!}\,\underset{\substack{\uparrow \\ k_2}}{(n-k)!}} \quad \alpha$$

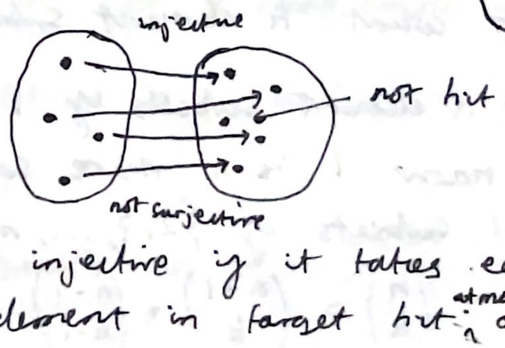
Multinomial Theorem

Let n be a positive integer

$$(a_1 + \dots + a_r)^n = \sum_{k_1+k_2+\dots+k_r=n} \binom{n}{k_1, k_2, \dots, k_r} a_1^{k_1} a_2^{k_2} \dots a_r^{k_r}$$



Pf: Same as binomial.



Functions

~~fKN→~~ $f: K \longrightarrow N$

Def: A function from K to N is injective if it takes each value of N $\boxed{\text{at most once}}$. [every element in target hit $\overset{\text{at most}}{\wedge}$ once].
[only exist if $|K| \le |N|$

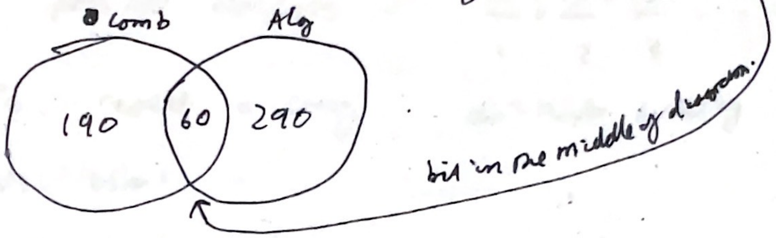
Def: A function from K to N is surjective if it takes each value of N $\boxed{\text{at least once}}$. Only exist if $|K| \geq N$ Bijective if both. $|K| = |N|$

Bijective ⟺ invertible. ⟺ two sets are the same size ⟺ proof = build a bijection.

Inclusion Exclusion

Eg suppose 250 students take combinatorics and 350 are taking Algebra I. How many are taking at least one.
Ans ∈ [350, 600] ⟺ How many taking both, + venn diagram.

hello ☺



250 + 350 − 60 = ans. ← to remove the double counting.

$|C \cup A| = |C| + |A| - |C \cap A|$

$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap C| - |B \cap C| - |A \cap B| + |A \cap B \cap C|$

In general:

$$\boxed{|A_1 \cup A_2 \cup \dots \cup A_n| = \sum_{\substack{I \subseteq \{1,\dots,n\} \\ I \neq \emptyset}} (-1)^{|I|-1} \left| \bigcap_{i \in I} A_i \right| \quad \leftarrow \text{all subsets}}$$

Pf: Pick a student and make sure picked only once.

I ♡ U Application: n! permutations of $\{1,\dots,n\}$. Eg $24314 \in S_5$.

$$\begin{matrix} 1 & 2 & 3 & 4 & \boxed{5} \\ 2 & 4 & 1 & 3 & \boxed{5} \end{matrix} \leftarrow \text{Fixed point} \qquad \begin{matrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 4 & 5 & 3 & 1 \end{matrix} \leftarrow \text{derangement = 'no fixed point'}$$

How many derangements of $\{1,\dots,n\}$ are there?

Soln:
- Try count not derangements!
  - 1st column could match] or 1 or 2 could match ⟹ unions!
  - 2nd column could match

Let A_i = set of permutations s.t. i is a fixed point $(\sigma(i) = i)$ $1 \le i \le n$.

derangements is $n! - \left|\bigcup_{i=1}^{n} A_i\right|$

$|A_1| = (n-1)!$ [n−1 remaining elements] ⟹ $|A_i| = (n-1)!$

$|A_i \cap A_j| = (n-2)!$

⋮

so $|A_1 \cup \dots \cup A_n| = \sum_{\substack{I \subseteq \{1,\dots,n\} \\ I \neq 0}} (-1)^{|I|-1} (n - |I|)!$

Combinatorics

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counting derangements, ⟺ If I randomly redistribute all of room's coats,
$\mathbb{P}$(nobody gets their coat back)

permutations that are not derangements,

$$= \sum_{\substack{I \subseteq \{1,\dots,n\} \\ I \neq \emptyset}} (-1)^{|I|-1} (n-|I|)!$$

$$\binom{n}{j} = \frac{n!}{j!(n-j)!}$$

$$= \sum_{j=1}^{n} (-1)^{j-1} \binom{n}{j} (n-j)!$$ [counting all subsets of size j]

[illegible]

so number of derangements of $\{1,\dots,n\}$ $= n! - \sum_{j=1}^{n} (-1)^{j-1} \frac{n!}{j!}$

$$= \sum_{j=0}^{n} (-1)^{j} \frac{n!}{j!}$$

$$= n! \sum_{j=0}^{n} \frac{(-1)^j}{j!}$$

so $\mathbb{P}$(nobody gets their own coat) $= \sum_{j=0}^{n} \frac{(-1)^j}{j!} \longrightarrow \frac{1}{e}$ as $n \to \infty$.

New counting Problem

How many ways are there to distribute k cookies among n people?

Eg $k=3$, $n=3$

- Give all to 1 person (3)
- ~~Gives 2 to p1, 1 to p2,~~
- Give 2/1/0   3 choices for 2, 2 choices for 1; (6)
- Give 1/1/1 → (1) →

need to show injectivity & surjectivity

so 10 choices in total. lets draw them.

Sequences of 3 stars and two bars are in bijection with distributions of cookies

$\underbrace{***}|\underbrace{\ }|\underbrace{\ }$

$\underbrace{**}|\underbrace{*}|\underbrace{\ }$

$*|*|*$

$|*|**$

3 stars, 2 bars. ~~[illegible]~~ 5 symbols in total.
so sequences of k ~~[illegible]~~ stars and n-1 bars.
To count sequences. 5 symbols choose which are stars.
k+n-1 symbols. How many ways to choose n-1 bars.
or stars.

$$C_{n,k} := \binom{k+n-1}{n-1} = \binom{k+n-1}{k} \swarrow \text{answer.}$$

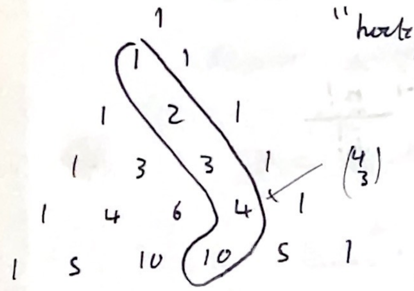
$$C_{n,k} = \sum_{i=0}^{k} C_{n-1,k-i}$$

cookies to first person, 1, 2, 3, ..., k ↑ all cookies to first

↑ once chosen 1st person, now have k-i cookies to distribute among the n-1 people remaining

$C_{n-1,0}$

$$\binom{k+n-1}{k} = \binom{k+n-2}{k} + \binom{k+\overset{n}{\cancel{2}}-3}{k-1} + \binom{k+n-4}{k-2} + \dots + \binom{n-\cancel{}2}{0}$$



"hockey ~~stick~~ stick identity" = pattern comes from combinatorics.

New Problem

Given a finite set $\{1, \dots, n\}$, a set partition of $\{1, \dots, n\}$ is a division into non-overlapping non-empty subsets

How many set partitions of $\{1, \dots, n\}$ into k parts are there?

Example

$n=4$, $k=2$.

sizes: 1/3 : $\{1\}, \{2,3,4\}$ $\{3\}\{1,2,4\}$ $\{2\}, \{1,3,4\}$ $\{4\}\{1,2,3\}$

sizes: 2/2 : $\{1,2\}\{3,4\}$ $\{1,4\}\{2,3\}$ $\{1,3\}\{2,4\}$

so answer is 7.

Def: The sterling number $S(n,k)$ is the answer to this question.

Combinatorics

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$[n] = \{1, \dots, n\}$

Ex divide $[4]$ into 2 parts.

compute $S(n,k) = \#$ set partitions of $[n]$ into k parts.

$S(100, 20)$ is too hard for a computer! Need to be clever.

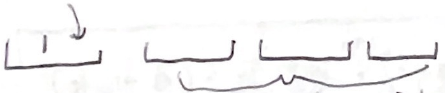
$\boxed{\lfloor 1 \rfloor \lfloor 234 \rfloor}$ $\lfloor 3 \rfloor \lfloor 124 \rfloor$ $\lfloor 12 \rfloor \lfloor 34 \rfloor$ $\lfloor 14 \rfloor \lfloor 23 \rfloor$

$\lfloor 2 \rfloor \lfloor 134 \rfloor$ $\lfloor 4 \rfloor \lfloor 123 \rfloor$ $\lfloor 13 \rfloor \lfloor 24 \rfloor$

Break problem down, treat 1 as special.

①

j other elements



[1 has to be in another → how are the remaining elements distributed?

$$\sum_{j=0}^{n-k} \binom{n-1}{j} S(n-(j+1), k-1) = S(n,k)$$

take out 1, n−1 elements left, then pick k.

used up k+1 elements so n−(k+1) elements

~~j = n−k+1, j+1 = [illegible]~~

If $j > n-k$, # elements not with 1 would be $n-(j+1)$

② want: $S(n,k)$ to $S(n-1,k)$ relation.

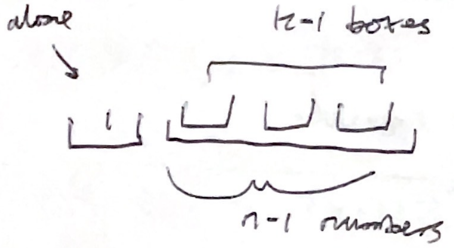
$\lfloor 23 \rfloor$ $\lfloor 47 \rfloor$ $\lfloor 5,10 \rfloor$ $\lfloor 8,9 \rfloor$ $\lfloor n \rfloor$

Q: How many set partitions of $[n]$ give this one when I delete one? A: k.

$k \cdot S(n-1,k)$ ← number of set partitions after deleting 1. doesn't count them all.

What have we missed? when one is alone. Missed these set partitions.

- Defined a function from set partitions of $[n]$ into k parts where 1 is not alone to the set of partitions of $\{2, \ldots, n\}$ into $k-1$ parts [deleate 1]
- k elements map from set 1 to set 2. Each has a size k.
- ~~Eg~~ If $\{1\} \to$ removed, need to count when one is alone.



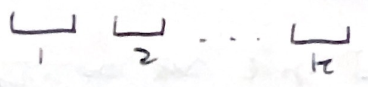
$\underline{A}: S(n-1, k-1)$

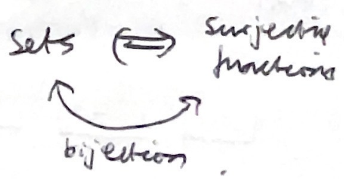
Summary: $\boxed{S(n,k) = S(n-1,k-1) + k\,S(n-1,k)}$

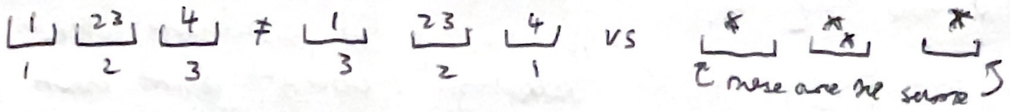
	k = 1	2	3	4	5	6
n = 1	1					
2	1	1				
3	1	3	1			
4	1	7	4	1		
5	1	15	25	10	1	
6	1	31	90	65	15	1

boxes before weren't labelled.

How about







Summary: $k!\,S(n,k)$ is # set partitions of $[n]$ with k labelled parts.

Think of this as a function from $[n]$ to $[k]$. This is surjective.

Process is invertable. Bijective,

let A_i = { functions $[n] \to [k]$ s.t. f does not hit i }

$\not\exists x \in [n]$ s.t. $f(x) = i$

So A_i is not surjective set of functions.

Compute:

$$|A_i| = (k-1)^n \leftarrow$$ k-1 choices for each of the 1,...,n inputs

$$|A_i \cap A_j| = (k-2)^n$$ functions that don't hit i or j

$\vdots$

$$\left|\bigcap_{i\in I} A_i\right| = (k-|I|)^n$$

$$|A_1 \cup \dots \cup A_k| = \sum_{\substack{I \subseteq [k] \\ I \neq \emptyset}} (-1)^{|I|-1} (k-|I|)^n$$

$$= \sum_{j=0}^{k} (-1)^{j-1} (k-j)^n \binom{k}{j}$$

$$= n^k - k!\, S(n,k)$$

from bijection set up.

$$\boxed{S(n,k) = \frac{1}{k!} \sum_{j=0}^{k} (-1)^j (k-j)^n \binom{n}{k}}$$

take $S(n,k)$ counting problem & tackled in in several ways. Translated counting problems, set up bijections

Def: Bell numbers = # set partitions of $[n]$ into any number of non-empty parts.

$$B_n = \sum_{k=1}^{n} S(n,k).$$

Generating Functions

$B_n = 1, 1, 2, 5, 15, 52, 203, 877, \ldots$

Power Series:

$$\frac{1}{0!} + \frac{1}{1!}x + \frac{2}{2!}x^2 + \frac{5}{3!}x^3 + \frac{15}{4!}x^4 + \ldots = e^{(e^x - 1)}$$

Remark: $S(n, k)$

(1) $n = 0$? ~~no~~ sets $\begin{cases} 0 & , k > 0 \\ 1 & , k = 1 \end{cases}$

(2) $k = 0$? $\begin{cases} 0 & ~~n \neq 0~~ \; n > 0 \\ 1 & n = 1 \end{cases}$

$$B_0 = 1 = \sum_{k=0}^{0} S(n, k) = S(n, 0) = 1$$

Def: Let $a_0, a_1, \ldots$ be a sequence. The exponential generating function of $a_0, a_1, \ldots$ is

$$\boxed{A(x) = \sum_{n=0}^{\infty} \frac{a_n}{n!} x^n}$$

[may not be a function]

Def: The ordinary generating function of $a_0, a_1, \ldots$ is

$$\sum_{n=0}^{\infty} a_n x^n$$

Ex $a_{n+1} = 4a_n - 100$.

OGF: $50 + 100x + 300x^2 + 1100x^3 + 4300x^4 + \ldots = A(x)$

$\overset{\times 4}{\Rightarrow}$ $200 + 400x + 1200x^2 + 4400x^3 + \ldots = 4A(x)$

$0 + 200x + 400x^2 + 1200x^3 + \ldots = 4xA(x)$

$4xA(x) - A(x) = -50 + 100x + 100x^2 + 1000x^3 + \ldots$

$= -150 + 100 \sum_{n=0}^{\infty} x^n$

$\cancel{A(x) = \frac{100}{(1-x)} - \frac{150}{}} = -150 + \frac{100}{1-x}$

$$\boxed{A(x) = \frac{\frac{100}{1-x} - 150}{4x - 1}}$$

$$A(x) = \frac{150x - 50}{(4x-1)(1-x)}$$

portial fractions

$$\frac{a}{4x-1} + \frac{b}{1-x}$$

$$\boxed{\frac{100/3}{1-x} - \frac{\overset{50}{\cancel{400}}/3}{4x-1} = A(x)} \Rightarrow \boxed{A(x) = \frac{100/3}{1-x} + \frac{50/3}{1-4x}}$$

$$\cancel{\frac{100}{3}(1+x+x^2+\dots) - \frac{100}{3}(1+4x+(4x)^2+\dots)}$$

$$\cancel{a_n = \frac{100}{3} - \frac{100}{3}\cdot 4^n}$$

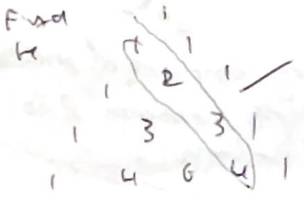
so $A(x) = \frac{100}{3}\sum_{n=0}^{\infty} x^n + \frac{50}{3}\sum_{n=0}^{\infty}(4x)^n$

so coefficient of $x^n = \frac{100}{3} + \frac{50}{3}\cdot 4^n$

Some diagonal of pascal

Ex: $\binom{n}{k}$?

nowhere'd sequence. $A(x) = \sum_{n=0}^{\infty}\binom{k+n}{k}x^n$



$$= \sum_{n=0}^{\infty}\frac{(k+n)!}{k!\,n!}x^n$$

$$A'(x) = \sum_{n=1}^{\infty}\frac{(k+n)!}{k!\,(n-1)!}x^{n-1}$$

check: $(1-x)A'(x) = (k+1)A(x)$

$$\Leftrightarrow \int\frac{A'(x)}{A(x)} = \int\frac{k+1}{1-x}$$

$$\Rightarrow \log(A(x)) = -(k+1)\log(1-x) + c$$

$$A(x) = \cancel{B(1-x)e^{-(k+1)}}\ B(1-x)^{-(k+1)}$$

$\boxed{A(0)=1} \Rightarrow B = 1 \Rightarrow \boxed{A(x) = \frac{1}{(1-x)^{k+1}}}$

$\binom{k}{k} = 1$

$$A(x) = \binom{k}{k} + \binom{k+1}{k}x + \binom{k+2}{k}x^2 + \dots$$

$$A(0) = \binom{k}{k} = 1$$

Second proof that $A(x) = \frac{1}{(1-x)^{k+1}} =$

$(1+x+x^2+\dots)(1+x+x^2+\dots)\dots\dots(1+x+x^2+\dots)$ $k+1$ times

what's the coefficient of x^n?

Take x^2: x^2 in 1st and 1 everywhere else

x in 1st x in 2nd and 1 everywhere else

→ give them bars.

[$k+1$ factors/ people]

$*||**||$ coefficient of x^n is

$1 \to 1$
$2 \to 0$
$3 \to 2$
$4 \to 0$
$5 \to 0$

$$\binom{(k+1)+(n-1)}{n} = \binom{k+n}{n} = \binom{k+1}{n}_n \quad \begin{matrix} \text{people} \\ \text{cookies} \end{matrix} \Big]$$ use previous lecture

coefficient of x^n in $\frac{1}{(1-x)^{k+1}}$ is $\binom{k+n}{k}$

$$\iff \frac{1}{(1-x)^{k+1}} = \sum_{n=0}^{\infty} \binom{n+k}{k} x^n = A(x)$$

$$\boxed{\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\binom{k+n}{k} x^n y^n = \frac{1}{1-x-y}}$$

Generating functions with combinatorics $\iff$ analysis

e^{e^z-1}

~~see~~ studying combinatorial problems with generating functions & do hard analysis

claim: $B_n = \#$ set partitions of $[n]$

said $\sum_{n=0}^{\infty} \frac{B_n}{n!} x^n = e^{(e^x-1)} = A(x)$

Pf: use $B_n = \sum_{k=0}^{n-1}\binom{n-1}{k} B_k$

$$A'(x) = \sum_{n=1}^{\infty} \frac{B_n}{(n-1)!} x^{n-1} = \sum$$

$$= \sum_{n=1}^{\infty}\sum_{k=0}^{n-1}\binom{n-1}{k} B_k \frac{x^{n-1}}{(n-1)!}$$

$$= \sum_{k=0}^{\infty}\sum_{n=k+1}^{\infty}\binom{n-1}{k} B_k \frac{x^{n-1}}{(n-1)!}$$

$m = n-(k+1)$
$n = m+k+1$

$$= \sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\binom{m+k}{k} B_k \frac{x^{m+k}}{(m+k)!}$$

$m = n-(k+1)$
$n = m+k+1$

$$= \cancel{\sum_{k=0}^{\infty}\sum_{n=k+1}^{\infty}\binom{n-1}{k} B_k \frac{x^{n-1}}{(n-1)!}}$$

$$= \sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\frac{1}{k!\,m!} B_k x^m x^k$$

$A(0) = B_0$

$$= \left(\sum_{k=0}^{\infty}\frac{B_k}{k!} x^k\right)\left(\sum_{m=0}^{\infty}\frac{x^m}{m!}\right)$$

1 partition of the empty set.

$$= A(x)\cdot e^x \implies A'(x) = A(x)e^x$$

$$\implies \log(A(x)) = e^x + c$$

$B_0 = 1$

$c = -1$

$$\implies A(x) = e^{e^x-1}$$

Integer Partitions

Def: An integer partition is a way of writing a number n as a sum of positive integers.

e.g $n = 8$, $6+2$, $2+2+2+1+1$, 8, etc. Don't care about the order.

[order does not distinguish partitions, otherwise its a cookie distribution question].

Def: $p(n)$ is the number of integer partitions of n.

$p_k(n)$ is # of partitions of n into exactly k parts.

$p^{odd}(n) = \#$ partitions into odd parts etc.

Questions
- Formula?
- Generating function?
- Recursion?
- How to compute $p(100)$?

$p(n) = 1, 2, 3, 5, 7, 11, 15, 22, \dots$

$\underline{1} \ \underline{1} \ \underline{2} \ \underline{2} \ \underline{4} \ \underline{4} \ \underline{7} \ \underline{8}$

$p(8) = 8, 7+1, 6+2, 6+1+1, \dots, 3+3+2$ [Start with largest & only go smaller]

Recursion $\displaystyle\sum_{a=1}^{8} p^{\le a}(n-a)$ ← partitions of $n-a$ all of sum at most a.

↑ largest entry

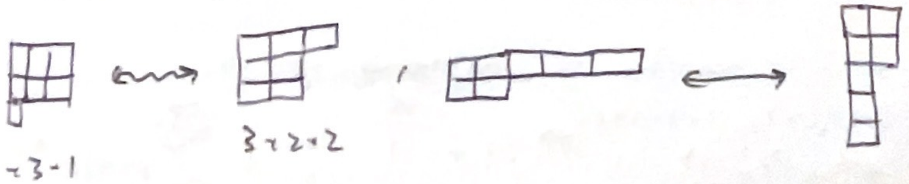
Notation: $3+3+1 \rightsquigarrow$  'young diagram' ← at most k rows

(# partitions of n into at most k parts) = (# number of partitions of n into parts that are at most k) ← at most k columns.

e.g $n = 7$, $k = 3$

$|\{7\cancel{6}, \overset{6}{\cancel{3}}+1, 5+2, 5+1+1, 4+3, 4+2+1, 3+3+1, 3+2+2\}| = 8$

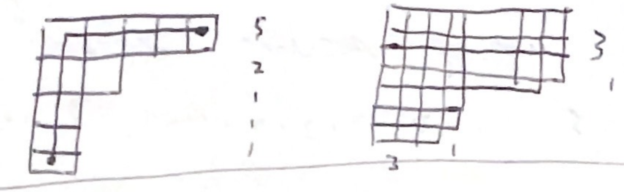
$|\{3+3+1, 3+2+2, 3+2+1+1, 3+1+1+1+1, 2+2+2+1, 2+1+1+1+1,$
$2+1+1+1+1\cancel{0}, 1+\dots+1\}| = \#8.$



2+2+1+1+1

Prop: (# partitions of n into distinct odd parts)
= (# self conjugate partitions of n)

$n=10$: ~~7,3~~ $7+3$, $9+1$,



Given a self conjugate young diagram, we can measure its 'Hook lengths'  check:

resulting partition

(a) is a partition of n ✓ [all boxes filled

(b) all parts distinct ✓ [length decreases by two each time]

(c) all parts are odd ✓

(d) bijection ✓ [algorithm for inverse map]

Prop: (# partitions of n into distinct parts) =
= (# partitions of n into odd parts)

Eg $n=8$: $7+1$, $5+3$, $5+1+1+1$, $3+3+1+1$, $3+1+\dots+1$, $1+\dots+1$ 6 odd parts

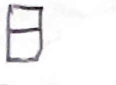
$7+1$, $6+2$, $5+3$, $5+2+1$, $4+3+1$, ~~2~~ 8 ← distinct parts

Pf: consider $(1+x+x^2+\dots)(1+x^2+\boxed{x^4}+\dots)(1+x^3+x^6+\dots)\dots$

coefficient of x^n? Is it well defined?

x^2: x^2 from 1st, 1 from rest ... well defined to compute.

x^{100}, only need to consider the 1st 100 terms. checked!



2 in 1st column, nothing else

$4+4+4+2+2+1 = 17$

1st factor: $x = (x^1)^1$

2nd factor: $(x^2)^2$

3rd factor: 1

4th factor: $(x^4)^3$

x^{17} → lots of ways.

~~$\prod_{k=1}^{n} \frac{1}{1-x^k} = \sum_{k=0}^{n} p(k)x^k$~~

correspondence:

coefficient of x^n is number of partitions of n.

4, $3+1$, $2+1+1$, $1+1+1+1$

$1\cdot1\cdot1\cdot x^4$, $x^3\cdot 1$, $x^2\cdot(x^1)^2$, $(x^1)^4$ $\Rightarrow$ $\boxed{1+\prod_{i=1}^{\infty}\frac{1}{1-x^i} = \sum_{n=1}^{\infty} p(n)x^n}$

Formalised, $n =$ 1 partition of n

a_1 1s, a_2 2s, a_3 3s, …

choose

x^{a_1} in 1st factor

$(x^{a2})^{a_2}$ in 2nd factor

$(x^3)^{a_3}$ in 3rd factor

$\vdots$

$$\Rightarrow \quad 1 + \sum_{n=1}^{\infty} p(n) x^n = \prod_{i=1}^{\infty} \frac{1}{1-x^i}$$

On the other hand,

$$(1 + x + x^2 + \dots)(1 + x^3 + x^6 + \dots)(1 + x^5 + x^{10} + \dots) \dots$$

is a partition of n into odd parts.

$$1 + \sum_{n=1}^{\infty} p^{\text{odd parts}}(n) x^n \cancel{= \prod_{i=1}^{\infty} \frac{1}{1-x}}$$

$$(1+x)\underbrace{(1+x^2)}_{\text{can't have 2 2s}}(1+x^3)\underbrace{(1+x^4)}_{\text{only 0 4s or 1 4}} \dots = \text{number of partitions into distinct parts.}$$

← odd parts.

$$\left(\frac{1}{1-x}\right)\left(\frac{1}{1-x^3}\right)\left(\frac{1}{1-x^5}\right) \dots$$

$$\left(\frac{1+x}{1-x^2}\right)\left(\frac{1+x^3}{1-x^6}\right)\left(\frac{1+x^5}{1-x^{10}}\right) \dots$$

Combinatorics - week 4 25/10/22

Proposition: (# partitions of n into odd parts) = (# partitions of n into distinct parts)

$$\boxed{1 + \sum_{n=1}^{\infty} p(n) x^n = \prod_{i=1}^{\infty} \frac{1}{1-x^i}}$$

last time.

← picks all the odds

$$\sum_{n=1}^{\infty} p^{\text{odd parts}}(n) x^n = \left(\frac{1}{1-x}\right)\left(\frac{1}{1-x^3}\right)\left(\frac{1}{1-x^5}\right)\dots$$

$$\sum_{n=1}^{\infty} p^{\text{distinct}}(n) x^n = (1+x)(1+x^2)(1+x^3)\dots$$

↖ picks all them once.

$$= \left(\frac{\cancel{1-x^2}}{1-x}\right)\left(\frac{\cancel{1-x^4}}{\cancel{1-x^2}}\right)\left(\frac{\cancel{1-x^6}}{1-x^3}\right)\left(\frac{\cancel{1-x^8}}{\cancel{1-x^4}}\right)$$

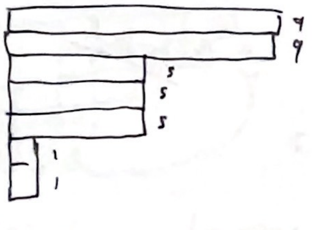
$$= \frac{1}{(1-x)(1-x^3)(1-x^5)\dots}$$

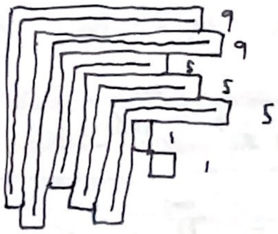
so two generating functions equal so

$$p^{\text{odd parts}}(n) = p^{\text{distinct}}(n)$$ ∎

Bijective proof:

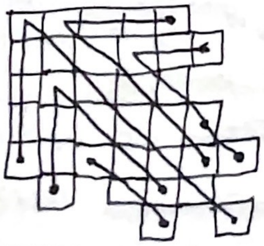
Sketch: Partition 35 into odd parts 9+9+5+5+5+1+1=35





stack them on top, bending around.

or



11 + 8 + 7 + 6 + 3

bijection between these 3 ways of looking at this.

$(1+x)(1+x^2)(1+x^3)(1+x^4)$

x^4: 0 1s, 0 2s, 0 3s, 1 4

1 1s, 0 2s, 1 3, 0 4s

Observe: $A(x) = \underbrace{\prod_{i=1} \frac{1}{1-x^i}}_{\text{generating function of partition numbers}}$ consider the inverse?

$$B(x) = \prod_{i=1} (1 - x^i)$$

$$A(x) \cdot B(x) = 1$$

$$B(x) = 1 - x - x^2 + x^5 + x^7 - x^{12} - x^{15} + x^{22} + x^{26} - x^{35} - x^{40} + \dots$$

$A(x) \cdot B(x) = 1$

$\Rightarrow \left(1 + p(1)x + p(2)x^2 + \dots\right)\left(1 - x - x^2 + x^5 + x^7 - x^{12} - x^{15} + \dots\right) = 1$

coefficient of x^n is:

$$\underbrace{p(n) \times 1 - p(n-1) \times 1 - p(n-2) + \underbrace{\quad}_{\text{no } x^3} + p(n-5) + p(n-7) \cdot - p(n-12)}_{} = 0$$

$A(x)B(x) = 1$ so <u>this</u> is equal to zero.

Rearranging, [Euler's pentagonal number theorem]

$$\boxed{p(n) = p(n-1) + p(n-2) - p(n-5) - p(n-7) + p(n-12) + \dots}$$

0 1 2 3 4 5 6
1, 1, 2, 3, 5, 7, 11, 15

$p(6) = p(5) + p(4) - p(1)$

Britch

$\boxed{B(x) = 1 - x - x^2 + x^5 + x^7 - x^{15} - \dots}$ <u>Pf</u>: ?

Recall, $1 + \sum_{i=1}^{\infty} p^{\text{distinct}}(n) x^n = \prod_{i=1}^{\infty} (1 + x^i)$

But $B(x) = \prod_{i=1} (1 - x^i) \leftarrow$ coefficient of x^n is # partitions of n into distinct parts <u>BUT</u> if # parts is odd, count them as -1.

(same as counting partitions where a partition with k parts contributes $(-1)^k$ to coefficient of x^n.)

$\Leftrightarrow$ (# partitions of n into even # parts − # partitions of n into odd # parts)

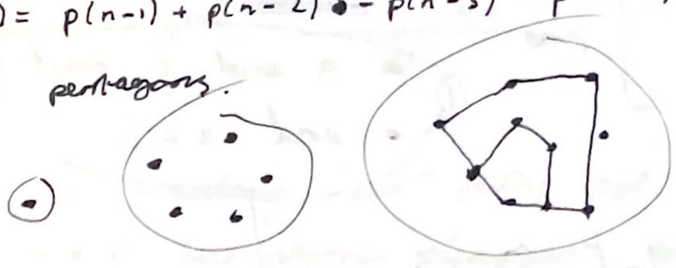
$n = 7$
$\overset{-1}{7}$, $\overset{+1}{6+1}$, $\overset{+1}{5+2}$
$\overset{+1}{4+3}$, $\overset{-1}{4+3+1}$
$\Rightarrow 1$

For most n, bijection between

Pentagonal Number thm

$p(n) = p(n-1) + p(n-2) - p(n-5) - p(n-7) + p(n-12) + \cdots$

nested pentagons. Formula ?

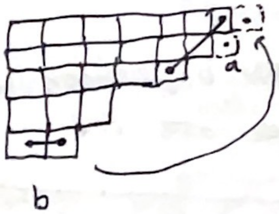


Reduced proof to a statement about even & odd # of distinct parts.

partitions of n with an odd number of distinct parts = # partitions of n with an even # distinct parts.

off by 1 in cases 1, 2, 5, 7, 12, ...

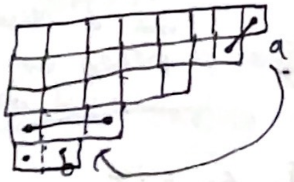
Example $7+6+5+3+2 = 23$



$b \leq a$ so can shift b to a on diagonal

⇒ so get a partition with 1 fewer part but they're all still distinct.

1 less part: odd # parts to even # parts [parity changed].



$a \leq b$ so shift a to b and get a new partition

⇒ added 1 part to partition (parity changed).

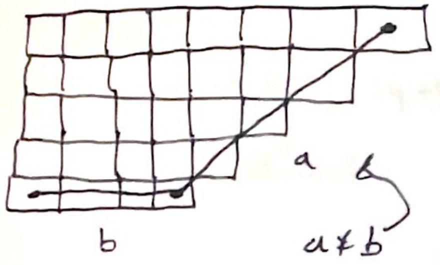
Bijection because it is its own inverse so a well defined bijective function. [relies on the fact that we have distinct parts]

operation

- If $b \leq a$, then move row b to lie diagonally next to a
- If $b > a$, then move diagonal a down to a new row below b

bijection between partitions of n with an odd & even number of distinct parts. when doesn't this work?

WHAT HAPPENS when b part of a ?



fine

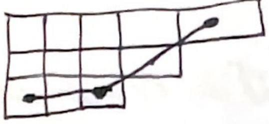
Bad if

① • a and b meet
• and a = b

problem points

② • a = b − 1
• a and b meet

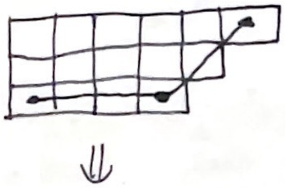
a = b 2



bad

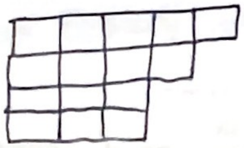
$5+4+3 = 7+6+5+4$ bad

6 + 5 + 4



bad

now

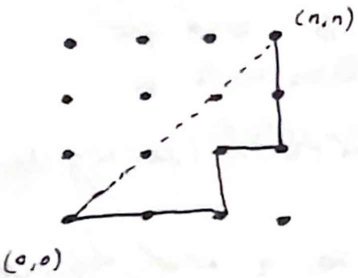


not distinct parts

If neither of the bad scenarios happen, bijection works. Turn bad scenarios into numbers & these are the numbers that don't work.

Catalan Numbers

Dyck path



Regular n+2 gons, count triangulations. [vertices are labelled]

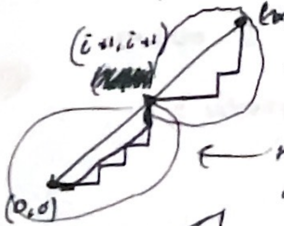
Find a recursion & then an explicit formula

$C_0 = 1,\ C_1 = 1,\ C_2 = 2,\ C_3 = 5,\ C_4 = 14$

Proposition: $C_{n+1} = \sum_{i=0}^{n} C_i C_{n-i}$

Pf: Given a Dyck path, look at when it first hits the diagonal (exclude origin). Will happen eventually. Break into stuff before that point ~~ob~~ and after that point.

so $\boxed{C_{n+1} = \sum_{i=0}^{n} C_{n-i} C_i}$



← choices for 2nd path $\boxed{C_{n-i}}$ choices

← need to not hit diagonal before] need 1st to be last to be 1 → Dyck paths from (0,0) to (i+1, i+1) that don't touch diagonal in bijection with Dyck paths from (1,0) to (i+1, i) [stay below diagonal] There are $\boxed{C_i}$

so

← Dyck path on diagonal below

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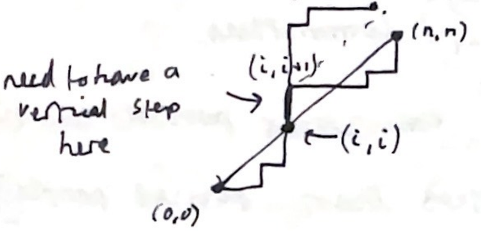
Generating Function:

$$\sum_{n=0}^{\infty} C_n x^n = \frac{1 - \sqrt{1-4x}}{2x}$$

Formula via counting "bad" paths that go above diagonal somewhere. $(0,0) \mapsto (n,n)$ but above diagonal. lets put these paths in bijection with something I can count more easily.



Apply a reflection

(i,i) is the first time path hits the diagonal.

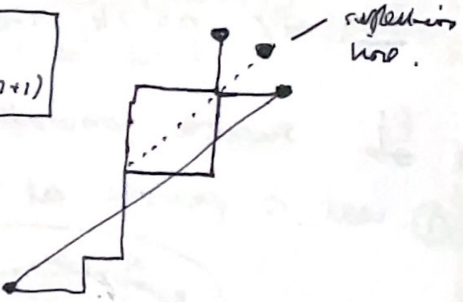
~~Reflected line~~ method:

- Bad path → reflect part from (i,i+1) → NE lattice path from $(0,0) \mapsto (n-1,n+1)$
- ~~Reflect path~~

bijection between these sets.

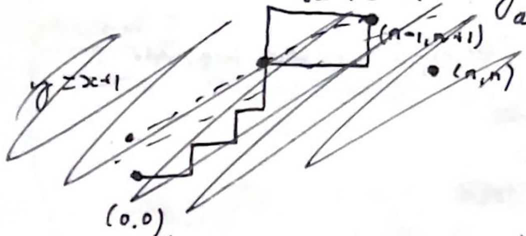
$\binom{2n}{n-1}$ of these

[assignment 1] ~~2n path~~ sequence of length 2n



bijection: reversible

reflect back along another line.



$$\left(\begin{array}{c}\#\text{ Dyck paths from}\\(0,0)\text{ to }(n,n)\end{array}\right) = \left(\begin{array}{c}\#\text{ NE lattice}\\\text{paths from}\\(0,0)\text{ to }(n,n)\end{array}\right) - \left(\begin{array}{c}\text{bad}\\\text{paths}\end{array}\right)$$

$$C_n = \binom{2n}{n} - \binom{2n}{n-1}$$

↑ path of length 2n, choose n moves] assignment 1

simplify $\boxed{C_n = \frac{1}{n+1}\binom{2n}{n}}$ $\sqcup$

Exteremal Combinatorics

- Let n, k · positive integers.
- Form as many k person committees as possible out of n people with constraint that any two share a member.

Strategy 1

↙ may be more

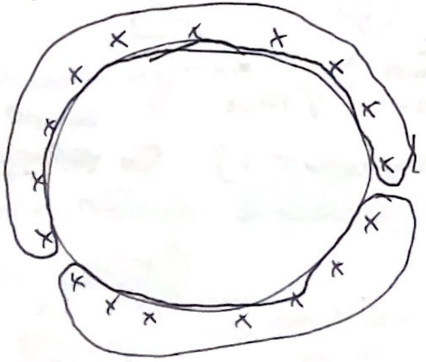
① Put person 1 on all committees. $\binom{n-1}{k-1}$ committees.

Observe: If $n < 2k$ just form all $\binom{n}{k}$ ~~committees~~ possible committees [forced to have overlap, way more committees than possible people.]

Thm: If $n \geq 2k$, then $\binom{n-1}{k-1}$ is the most committees you can form.

Pf: suppose committees $A_1, \dots, A_N$, $A_i \subset [n]$. $A_i \cap A_j \neq \emptyset$

① Seat n people at a circular table.



Is A_i seated ~~separately~~ seated consecutively.

$$X = \sum_{\substack{\text{all possible} \\ \text{seatings}}} (\# A_i\text{s sealed together})$$

~~[illegible]~~

Given i, how many seatings put A_i together?

$k!(n-k)!$ [k person committee] $k = |A_i| \ \forall i$

↗ fix committee ← fix people left.

$\Rightarrow X = N \cdot k!(n-k)!$

↑ There are N committees so sum from $i = 1 \dots N$

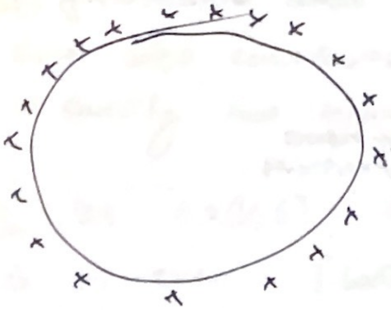
②

Combinatorics - week 5

Suppose $A_1, \dots, A_N$ are k element subsets of $[n]$ such that $\forall i,j$, $A_i \cap A_j \neq \emptyset$ [each committee shares at least 1 member].

Want: $N \leq \binom{n-1}{k-1}$

Pf:



compute $X = \sum_{\substack{\text{all possible}\\ \text{seatings}}} \# A_i\text{'s seated together.}$

$\sum$ [given a seating, how many A_is together]

Two ways to compute.

$N \geq 2k$
[otherwise, make $\binom{n}{k}$ committees] lots of committees.

Method 1:

- For each committee A_i, will always contribute 1 to sum. Showed A_i together in $\underbrace{k!}_{\text{seat members of committee}}\underbrace{(n-k)!}_{\text{seat everyone else.}}$ seatings.

- So summing over all N committees, $X = N \cdot k!(n-k)!$

Method 2:

- For a given ~~committee~~ seating, how many committees are together.

] can't have this as no-overlap.

can't have this. non-overlap.

$k=3$

is the only choice.

For each seating, we can have $\boxed{\text{at most } k.}$ use $n \geq 2k$ here. why? [think through small example] [shift each by one till you get a single overlap with the committee you started with]  [achieved if committees are all apart by shifting by one].

→ So adding up over all N seatings [up to rotation, fix 1

$\therefore \quad X \leq k \cdot \#\text{ seatings} = k \cdot (n-1)!$ [fix 1 and $n-1$ left to permute]

$\therefore \quad N \cdot k!(n-k)! \leq k(n-1)!$

$$\Rightarrow N \leq \frac{(n-1)!}{(k-1)!(n-k)!} = \binom{n-1}{k-1}$$

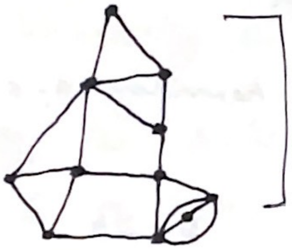
Proofs were messy at start & then got more streamlined

Graph Theory

Four Colour Theorem

- many wrong proofs
- proof was computer aided.

map of england: shape irrelevant. Graphs model connectivity / adjacency.



this is bad! — must name your vertices.

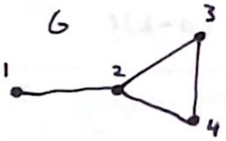
← ~~order~~ ordered pair of two sets.

Def: A graph, $G = (V, E)$ is the data of

1) a set V of vertices

2) a set E of unordered pairs of distinct vertices [edges]

Example:

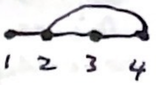


$V = \{1, 2, 3, 4\}$

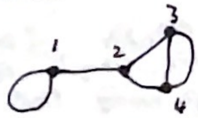
$E = \{\{1,2\}, \{2,3\}, \{3,4\}, \{2,4\}\}$

so infinite ~~a~~ graphs can exist. [not a finite structure]

can draw G in other ways:



Possible?



$E = \{\{1,2\}, \{2,3\}, \{2,4\}, \underbrace{\{3,4\}^2, \{1,1\}}_{\text{not valid according to def.}}\}$

But can loosen to define **this** as a graph [multigraph / graph].
and a simple graph is ~~one~~ defined ~~the~~ above
[no duplicates & loops].

Def: let $G = (V, E)$ be a graph. Let $v \in V$, the degree $\deg(v)$ of v is the number of edges of G that come out of [are incident to]. ~~degree~~ $\deg(2) = 3$

Graphs

- $v_1, v_2 \in V$ are adjacent / neighbours if $\{v_1, v_2\} \in E$

Prop: $\sum_{v \in V} \deg(v) = 2|E|$

Pf: each edge contributes 1 · let $G = (V, E)$ be a graph.
to exactly two terms of the sum.

Corollary: let $G = (V, E)$, the number of odd degree vertices of G is even. [look at sum, need even number of odd numbers to make it even].

Ex: the complete graph on n vertices, K_n has
$V = [n]$, $E = \{\{i, j\}, i, j \in [n], i \neq j\}$

K_5

K_4.

Ex: Path graph on n vertices, P_n.
$V = [n]$, $E = \{\{i, i+1\}, i \in \{1, \dots, n-1\}\}$

P_4.

Ex: Cycle graph on n vertices C_n
$V = [n]$, $E = \{\{i, i+1\}, i \in \{1, \dots, n-1\}\} \cup \{1, n\}$

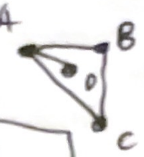
C_4

Ex: Random graphs with n vertices
$V = [n]$ include $\{i, j\}$ in E with some fixed probability $0 < p < 1$

Def: Two graphs $G = (V, E)$ and $G' = (V', E')$ are isomorphic if there is a bijection $\phi : V \longrightarrow V'$ s.t.
$\{v_1, v_2\} \in E \iff \{\phi(v_1), \phi(v_2)\} \in E'$

[same structure, just named vertices differently.

isomorphic graphs.



so $\phi(1) = 0$

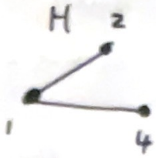
ϕ

$1 \leftrightarrow 0$

$2 \leftrightarrow A$

$3 \leftrightarrow B$

$4 \leftrightarrow C$



$(1,4) \notin E$

is a subgraph.

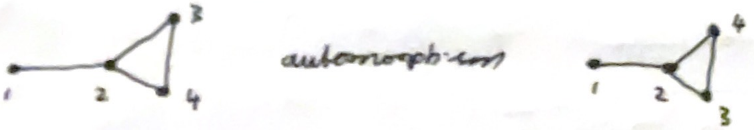
Def: let ~~G be a graph~~ $G=(V,E)$ be a graph, $H=(V',E')$ s.t. $V' \subset V$. then H is a subgraph of G if $E' \subset E$ [simple graph]

If $E' = \{\{v_1,v_2\} \in E \text{ s.t. } v_1,v_2 \in V'\}$ then H is an induced subgraph of G

Take subset of vertices and keep all the ~~sto~~ ~~st~~ edges remaining

Def: If $V'=V$, then H is a spanning subgraph [just delete edges].

Def: An automorphism of $G=(V,E)$ is an isomorphism from G to G.

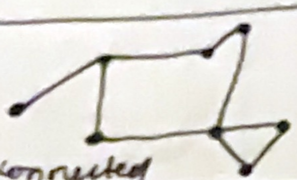


The automorphisms of G form a group $Aut(G)$ under compositions. [all the permutations of S_n that preserve the group structure].

If ~~is a~~ $G=(V,E)$ is a graph and H is isomorphic to a subgraph of G, we say G contains H.

Def: G is connected if for any $v_1, v_2 \in V$, G contains a path from v_1 to v_2.

walk allows for repeated vertices


connected

every v_1 to every v_2 path with no repeating vertices.

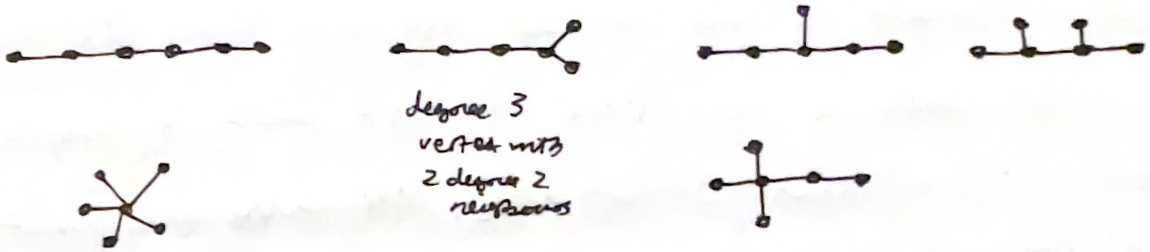
not connected.

connected components of G.

Def: A graph G is ~~[illegible]~~ acyclic if it contains no cycles

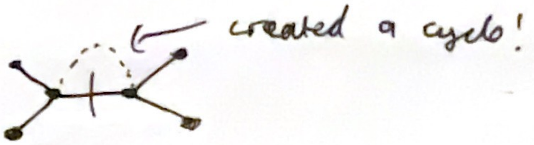
Def: A tree is a graph that is both connected & acyclic

[no root] Trees with 6 vertices



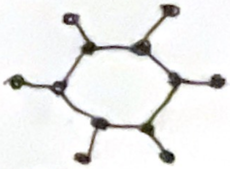
Thm: Let $G = (V,E)$ be a tree, ~~and suppose~~ Then $\forall e \in E$, deleting e yields a disconnected graph.

Pf: If not, $\exists e \in E$ s.t. $G \setminus \{e\}$ s.t. $G \setminus \{e\}$ is connected. ~~[illegible]~~ As connected, there is a path P from v_1 to v_2. Then P is also a path in G. adding e to P yields a cycle in G. $\Rightarrow$ G is not acyclic $\times$ to G a graph.



Thm: If $G = (V,E)$ is connected and deleting any edge disconnects G, then G is a tree

Pf: (formal in notes).



Trees are edge minimal connected graphs. Several ways to define them.

Idea: Inductively prove that $|V| = |E| + 1$ for trees.
For a tree, if you can find a degree 1 vertex, then you can delete it, show what's left is a tree and proceed by induction.

Lemma: Every finite tree with ≥ 2 ~~vertex~~ vertices has a leaf.

Pf: [walk without turning around till you hit a dead end].

Consider all paths in G. There exists a path of length at least 2, [at least 2 vertices]. All paths are finite.
Find a maximal path [~~[illegible]~~? the one with as many vertices as possible] well defined question.

Claim: endpoints are leaves.

If they weren't leaves, have two edges coming out of it, so can add $\Rightarrow$ makes a longer path ✗.

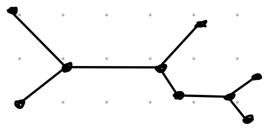
to maximal length.

Combinatorics, week 6

8/11/22

Thm: Every tree has a leaf

Pf: [strategy, consider a path in T that is as long as possible]



Consider a path in T that is as long as possible.

Endpoints of P are called v_1, v_2. If v_1 is not a leaf, there is an edge, not in P that is incident to v_1, say e.

Two cases for e: [let v_0 be the other endpoint of e]

① v_0 is not a vertex in P. Then, could add e and v_0 to P to get a longer path, contradicting P's maximal length.

② v_0 is in P, then we get a cycle:

$$v_0 \xrightarrow[P]{} v_1 \xrightarrow[\text{[not in P]}]{e} v_0$$

contradicting that T is a tree.

$\Longrightarrow$ v_1 (and v_2) are leaves $\Longrightarrow$ every tree has a leaf. [really, at least 2] □

Rmk: Infinite trees need not have leaves.

Corollary: If T is a tree with n vertices, then T has n-1 edges

Pf: Induction on n.

Base case: 1 vertice, no edges ✓

If T has n+1 vertices, choose a leaf v. Delete the leaf v & its edge. (this can't have introduced a cycle) The resulting graph is still connected and acyclic, and a tree with n vertices $\Longrightarrow$ n-1 edges [by induction hypothesis], so T had n edges when we add v back into the graph.

[converse of corollary]

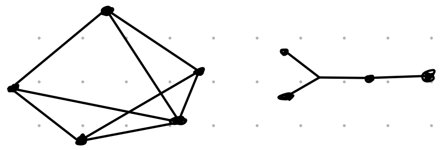
If G has n vertices, n-1 edges, and is connected, then G is a tree. $|V(G)| = n$, $|E(G)| = n-1$

Use the fact that trees are the edge minimal connected graphs. Tree is a connected graph s.t. if you delete any edge, then T is disconnected.

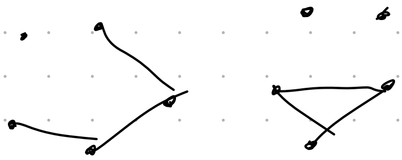
Pf: let G be a connected graph. Let H be a connected spanning [contains all the vertices] subgraph with as few edges as possible. [well defined as G finite]. so by 1st Thm proved about trees, H is a tree. H has all the vertices as H is a spanning subgraph $\Longrightarrow$ $\underbrace{|V(H)|}_{n} = \underbrace{|V(G)|}_{n}$. But as H is a tree, $\underbrace{|V(H)| = |E(H)| + 1}_{\text{tree has 1 more vertex than edges}}$

Started with n-1 edges, ended with n-1 edges, so H = G. [haven't actually deleted any edges] $\Longrightarrow$ G is a tree [as H is a tree] □

Rmk: "Spanning tree" of a connected graph $\iff$ connected spanning subgraph with as few edges as possible.



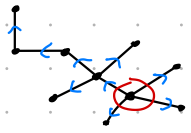
Rmk:



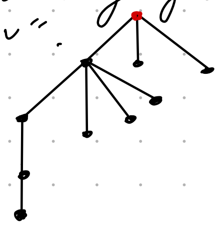
Adding an edge either joins two components or creates a cycle.

[connection between linear algebra & graph theory]

Rmk: Given a tree T & a vertex v, there is a unique way of assigning a direction to each edge s.t. all edges "point away from v".



[because our graph is acyclic, this is well defined]



Problem of Colouring graphs

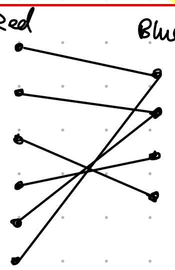
Given a graph G, how many colours & needed to paint the vertices so that adjacent vertices get different colours?

$\iff$

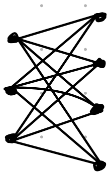
want to partition the vertices into k subsets so that each subset contains no pair of adjacent vertices. "Proper k colouring".

[all the red vertices would be one subset]

If a proper k colouring exists, then we say G is "k-colourable" or "k-partite". The minimum such k is $\chi(G)$, the vertex colouring number.

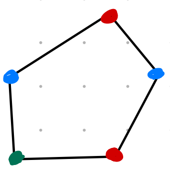


two-partite graph



complete-bipartite $K_{3,4}$

Thursday 10th November 2022

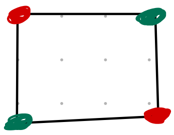


"Proper vertex 3-colouring of C_5"

$\chi(G) = 3$

[min number of colours needed]

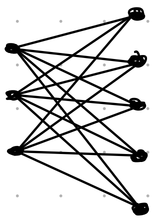
equally, partition V into subsets with no edges among the elements. "independent set of vertices"



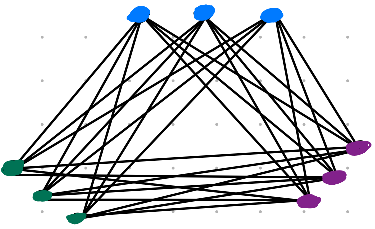
$$\chi(C_n) = \begin{cases} 2 & n \text{ even} \\ 3 & n \text{ odd} \end{cases}$$

Remark: $\chi(G)$ is NP-complete: noone knows an algorithm that is much better than just brute force.

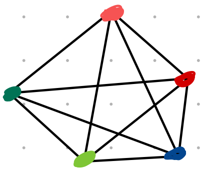
Recall: For $a, b > 0$, defined $K_{a,b}$



Def: For $a_1, \ldots, a_r > 0$ complete r-partite graph $K_{a_1,\ldots,a_r}$



What is the chromatic number of the complete graph?



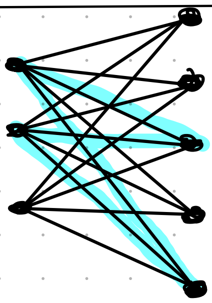
$$\boxed{\chi(K_n) = n}$$

For every choice of two colours, can find an edge between them

In a minimal vertex colouring $\chi(G)$ of G, any two colour classes are joined by an edge:

$$|E| \geq \binom{\chi(G)}{2} = \frac{\chi(G)(\chi(G)-1)}{2}$$

Using quadratic formula & rearranging, $\chi(G) \leq \frac{1}{2} + \sqrt{2|E| + \frac{1}{4}}$



Bipartite graphs

- Never have odd length cycles. [there & back]

Thm: A graph is bipartite $\iff$ it has no odd cycles.

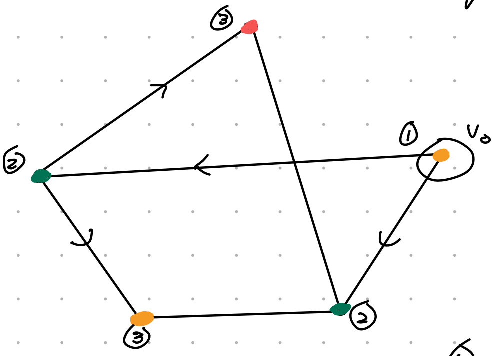
So no odd cycles $\implies$ can two colour the graph.

Pf: Assume G is a connected graph with no odd cycles. Every connected graph has a spanning tree. [a tree that uses all the vertices – keep deleting edges until graph is unconnected]. Let T be a spanning tree. Choose a root v_0 of T.

For each v, take the unique path in T from v to the root.

even length 🟩

odd length 🟧



- Find a tree [spanning]
- Pick a root
- go along edges of tree & colour.

Need to show this is a valid graph.

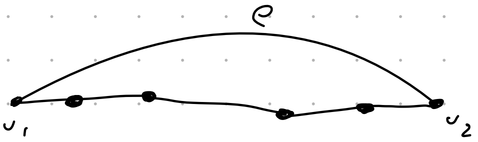
Need: For every edge of G, the endpoints are different colours. $e \in E(G)$

Cases:

① If $e \in E(T)$, the unique path from endpoints further from v_0 to v_0 is longer by 1 than the path from other endpoint to v_0

If $e \notin E(G)$, Suppose the endpoints v_1 and v_2 are same colour. Take the path P in T from v_1 to v_2 & add the edge e. [now a cycle]

Note: Along P, the colours alternate.



Since v_1 and v_2 are the same colour $\Rightarrow$ P has an odd number of vertices and adding edge e $\Rightarrow$ odd cycle. [but we assume graph has no odd cycles. ✗.

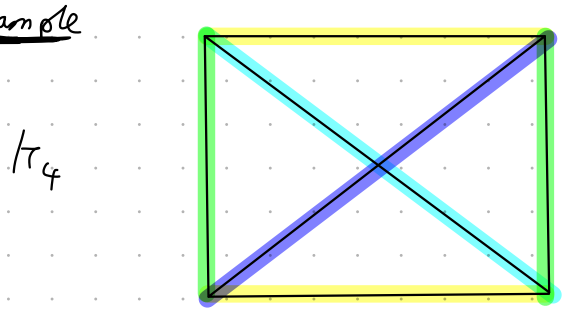
Edge Colouring

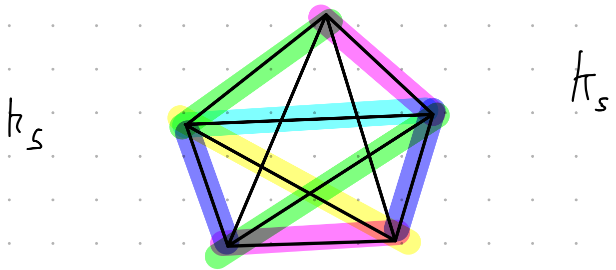
A proper edge-colouring of G is an assignment of a colour to each edge, such that if the two edges share a vertex, they are different colours.

"independent set of edges" $\Leftrightarrow$ no two share a vertex

Def: $\chi'(G)$, edge colouring number is the minimum number of colours needed for a proper edge colouring.

Example



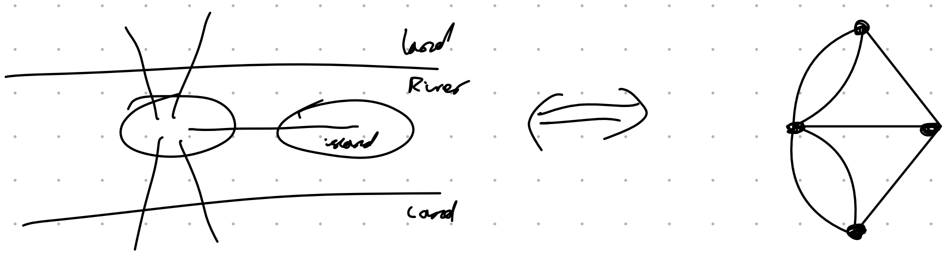


[Same as football teams tournaments]

Observe: $\chi'(G) \geqslant \max deg(G)$

Vizing's Thm: $\chi'(G) \leq \begin{cases} \max deg(G) \\ \max deg(G)+1 \end{cases}$

GRAPH TRAVERSAL PROBLEMS



Def: An Eulerian tour of G is a walk on G that uses every edge exactly once and ends where it starts.

Obs: If G has a vertex of odd degree, impossible!

Thm: (Euler 1789) If G is connected and $\deg(v)$ is even $\forall v \in V$, then G has an Eulerian tour.

Cases to worry about:

- arrive at a vertex v & there are no unused edges from v.
  - → If v is not the start, you enter & leave, even degree through so fine.
  - → If $v = v_0$ [start vertex], no unused edges only if $v = v_0$.
- Strategy: Start walking till you can't move anymore. If stuck, take original walk delete all edges & then start a detour.

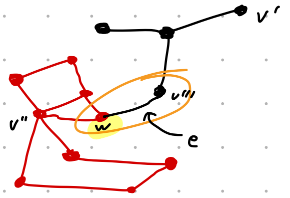
Pf: let W be the longest walk in G that doesn't repeat any edges. [hopefully this will have used all the edges].

Let v_0 be the ending vertex of W.

Cannot go further $\Rightarrow$ all edges at v_0 are in W. [even number of those].

Since $\deg(v_0)$ is even, v_0 is also the starting vertex of W. [otherwise can leave & enter]

W could be Eulerian walk $\Rightarrow$ suppose it's not.



If G has unused edges, let v' be in this edge.

Choose v'' in W, and take a path $v' \to v''$.

$\Rightarrow \exists$ vertex of W with an unused edge coming out of it.

We assumed W was of maximal length, but we can make a longer path.

New path is: start a v''', walk along e, follow W

$$v''' \xrightarrow{e} w \xrightarrow[\text{(some of it till } v_0)]{\text{around } W} v_0 \xrightarrow[\text{(bit after } v_0\text{. cycle so fine).}]{\text{rest of } W} w$$

$\Rightarrow W$ is not the longest such path $\Rightarrow$ contradiction. ∎

Def: A Hamiltonian cycle/tour is a walk that passes through each vertex exactly once & ends where it starts.

A Hamiltonian tour of a graph G is a spanning cycle in G (walk in G that visits every vertex exactly once, and finishes where it starts.

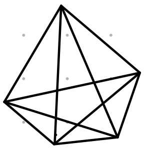
Eulerian Cycle	Hamiltonian Graph
• visit every edge exactly once Edge = Eulerian	• visit every vertex exactly once.

- Deciding if a graph is Eulerian is easy: even degree
- " " Hamiltonian is hard.

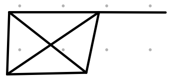
Examples

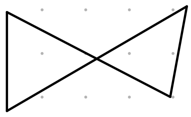
Need lots of connections

4! total Hamiltonian tours of K_5



Not Hamiltonian.

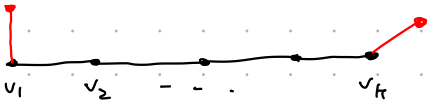




[one of the most elegant proofs in the module]

Thm: Suppose G is a simple connected graph with $n \geq 3$ vertices, such that every vertex has degree $\deg(v) \geq \frac{n}{2}$, then G has a Hamiltonian tour.

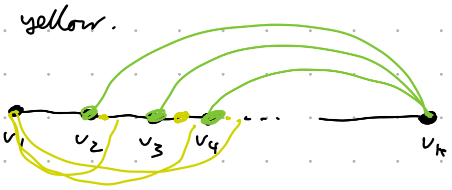
Pf: [Idea: need to construct a cycle. Will need to remember key ideas]
Let G be a simple graph with $n \geq 3$ vertices and every vertex has degree $\geq \frac{n}{2}$
Let P be a path in G of maximum length with vertices $v_1, \dots, v_k$



Since P is a path, we can't repeat vertices so $k \leq n$.
Note $\deg(v_1) \geq \frac{n}{2}$, $\deg(v_k) \geq \frac{n}{2}$ and all neighbours of v_1, v_k are in P as otherwise we would have a longer path.

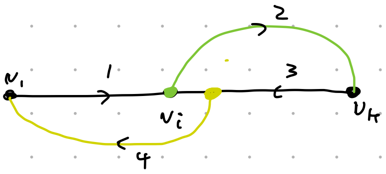
Since $\deg(v_k) \geq \frac{n}{2} \implies v_k$ has $\geq \frac{n}{2}$ neighbours in $v_1, \dots, v_{k-1}$, colour them green.

Since $\deg(v_1) \geq \frac{n}{2} \implies v_1$ has $\geq \frac{n}{2}$ neighbours in $v_2, \dots, v_k$, If v_{i+1} is a neighbour, colour v_i yellow.

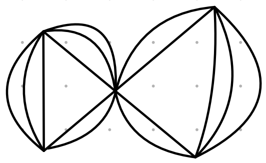


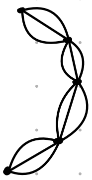
We didn't need to state G connected explicitly, it follows from G simple with $\deg(v) \geq \frac{n}{2}$ $\forall v$

We have given $\geq \frac{n}{2} + \frac{n}{2} = n \geq k \geq k-1$ colours to $v_1, \dots, v_{k-1}$ so at least one vertex must have both colours.



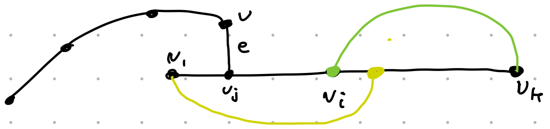
Note: "Simple" was necessary!





Consider the cycle C:
- $v_1 \longrightarrow v_i$ along P
- Then edge $v_i v_k$
- Then v_k to v_{i+1} backward along P
- Then edge v_{i+1} to v_1.

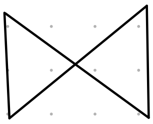
Need to show we haven't excluded any vertices in G from C. So claim C is a Hamiltonian cycle, so suppose it is not. Then there is a vertex v_0 not in G.
As G is connected, there is a path from v_0 to a vertex in P.
Then there is an edge e in this path between $v_j \in P$ and $v \notin P$



Start at v, travel along e to v_j, then travel around the cycle C in either direction, stopping just before we return to v_j. This is a path with $k+1$ vertices contradicting that P is the longest path.

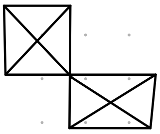
$\implies$ all vertices of G lie in $C \implies C$ is a Hamiltonian cycle.

Q: Could we improve on $\frac{n}{2}$?



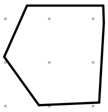
n = 5

$\deg(v) \geq 2$



n = 7

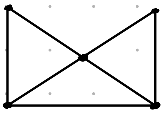
$\deg(v) \geq 3$

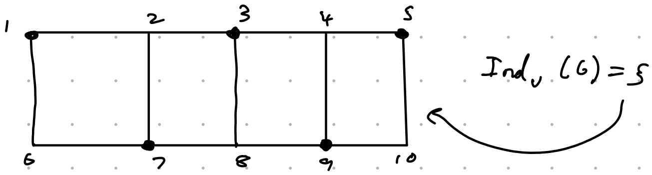


Matchings

17/11/22

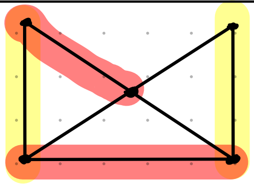
A set of vertices of a graph G is independent if there are no edges joining any two of them.

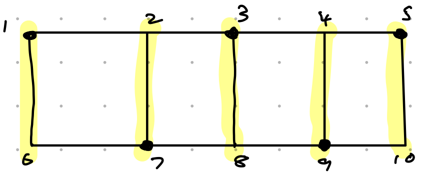




Def: The vertex independence number $\text{ind}_v(G)$ of a graph G is the maximum size of an independent set of vertices.

Def: a set of edges of G is independent / matching if no two share a vertex



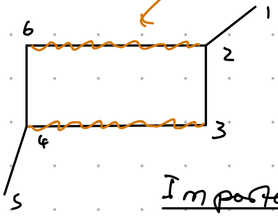


Def: The matching number $\text{ind}_E(G)$ is the maximum size of a matching.

Note: $\text{ind}_E(G) \leq \frac{|V(G)|}{2}$ [every edge uses two vertices]

Def: A matching is perfect if it has $\frac{1}{2}|V(G)|$ edges and we say G admits a perfect matching.

best we can do is two.

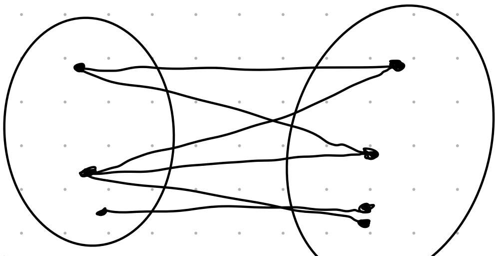


] no 3 matching
⇒ no perfect matching

Intuition: Computing
$\text{ind}_v(G)$ is hard,
$\text{ind}_E(G)$ is easier

Important special case

G is bipartite. $V(G) = A \cup B$, all edges join A and B.



A = jobs

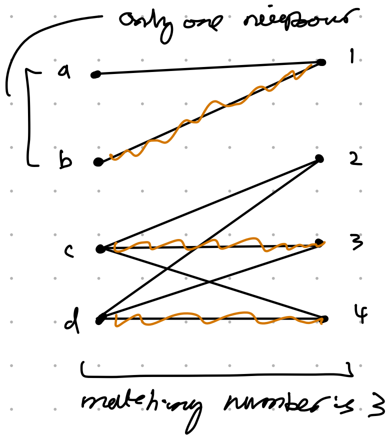
B = applicants

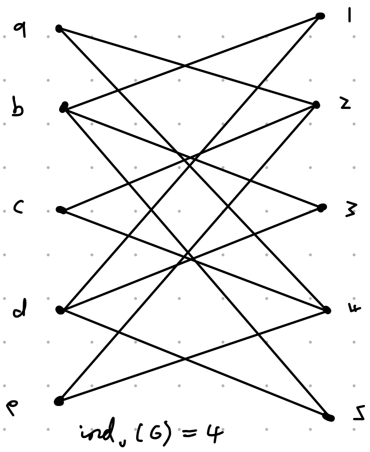
Given a bipartite graph, is there a matching that uses every vertex of A?

Note, if G is bipartite
$\text{ind}_E(G) \leq \min(|A|, |B|)$

For a perfect matching, we need $|A| = |B|$.

We say G has a matching of A if $\text{ind}_E(G) = |A|$ ⇒ every vertex in A is used in the matching.





$\text{ind}_v(G) = 4$

Problem is a, c and e only have 2 neighbours

necessary & sufficient condition for matching.
⇑

In order to have a matching of A [match everything in A], we must have that for every $S \subseteq A$, we have $|N(S)| \geqslant |S|$ [Hall's condition]

↑
Neighbours of $S = \{ b \in B : (s, b)$ is an edge for some $s \in S \}$

Thm: If G is a bipartite graph with $V(G) = A \cup B$ & G satisfies Hall's condition, there is a matching of A. [Hall's thm 1935]

Surprising useful theorem!

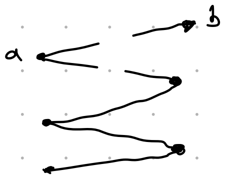
Pf: By induction on the size of A. $|A|$.

If $|A| = 1$, since $|N_G(A)| \geqslant 1$, G has an edge so there is a matching of A.

Now suppose that $|A| > 1$ and thm true for smaller A. Two possibilities

① For every non-empty proper subset $S \subsetneq A$, $|N_G(S)| \geqslant |S| \underbrace{+1}$ Stronger

In this case, pick an edge $e = (a, b)$ & remove e, and a and b (& all edges involving a, b) from G to form G'.

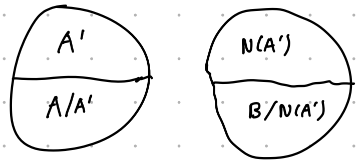


The new graph G' has $A' = A \setminus a$ and $B' = B \setminus b$
& for all $S \subseteq A'$, $|N_{G'}(S)| \geqslant |N_G(S)| - 1$ [removed at most b]
$\geqslant |S| + 1 - 1$
$= |S|$

So Hall's condition holds for G' so by induction, G' has a matching M for A'. $M \cup \{e\}$ is a matching for A in G.

② There is a proper non-empty subset $A' \subseteq A$ with $|N_G(A')| = |A'|$

Goal:



We will find matchings of A' & of A/A' that do not share vertices.

Since $|A'| < |A|$ by induction there is a matching of A'. The endpoints of this matching in B are a subset of $N_G(A')$ of size $|A'|$, so equals $|N_G(A')|$.

Now delete A' and $N_G(A')$ from G to get a new graph G''. This is bipartite with with partition $A \setminus A'$ & $B \setminus N_G(A')$

We claim G'' satisfies Hall's condition [our one tool, so better use it].

Fix $S \subseteq A \setminus A'$

$$|N_{G''}(S)| = |N_G(S \cup A')| - |N_G(A')|$$ [G & G'' don't overlap ,

$$\geq |S \cup A'| - |N_G(A')|$$

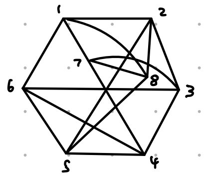
$$= |S| + |A'| - |A'|$$

$$= |S|$$ $\Rightarrow$ G'' satisfies Hall's condition, G'' smaller so by induction, there is a matching for $A \setminus A'$ in G''.

We have a matching of A' and $A \setminus A''$, don't share neighbours. Union of these two matchings gives a matching of A in G. ∎

Thm: Peterson (1891). Let $G = (V, E)$ be a graph where every vertex has degree $2k$ for some constant $k > 0$. Then G has a spanning subgraph that is a union of disjoint cycles.

no bipartite graphs. Care more about proof than statement.



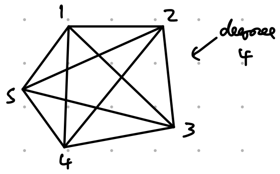
Pf: If G is not connected, we'll find the spanning cycle for each connected component separately $\Rightarrow$ assume G is connected.

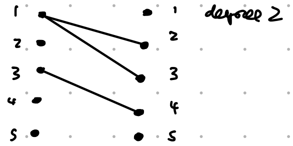
Since the degree of every vertex is even $V = \{v_1, \dots, v_n\}$ $\Rightarrow$ G is Eulerian.
$\Rightarrow$ There is an Eulerian tour W. It visits every edge once. [need a bipartite graph]

We now define a bipartite graph G' with vertices $A = \{a_1, \dots, a_n\}$, $B = \{b_1, \dots, b_n\}$ & an edge connecting a_i to b_j if (v_i, v_j) is an edge in the tour W. [no edge a_i, b_i as no loops]

In A4, will show a regular bipartite graph has a perfect matching, so need to show G' is regular (

Note: The degree of $a_i \in A$ is the number of edges (v_i, v_j) for some j in W.



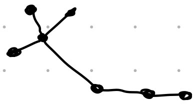


in k times
out k times

Tour: 1 2 3 4 5 1 3 5 2 4 1 ↗

This is k because the tour leaves v_i exactly k times
$\Rightarrow$ G' has a matching, translate matching back to edges of G, we find a subgraph where every vertex has degree 2, so is a union of cycles. ∎

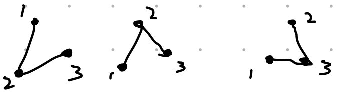
Cayley's Tree enumeration formula.



$\frac{7!}{3!}$

for n=7 labelled vertices, found 16807 labelled trees.

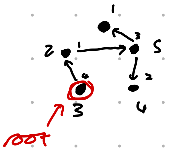
n=2	3	4	5	6	7
1	3	16	125	1296	16807



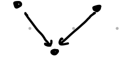
formula: $\boxed{n^{n-2}}$ non-trivial counting problem.

- Bijective proof of formula: trees with n vertices with n sequences of n-2 elements. messy
- linear algebra proof: [in lecture notes, also electrical circuits]
- Double counting: find some object you count in two different ways. [not clear why this is the right quantity to count].

Pf: let X denote the set of sequences of directed edges that when put together, make a directed rooted tree.



root is 3



this never happens in a directed rooted tree.] also care about the ordering of the edges.

way ① to find |X|:

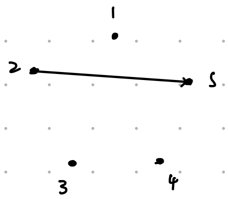
T_n = number of labelled trees with n vertices

$$|X| = T_n \cdot n \cdot (n-1)!$$

number of orderings of the (n-1) edges of the tree [choosing a sequence of length n-1 of directed edges].

number of ways to pick a root

way ② to find |X|:



chose an edge.

- 5 choices for where to start arrow
- 4 choices for where to end arrow

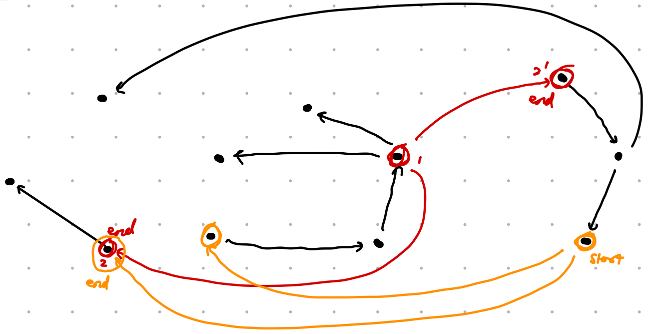
] ⟹ n(n-1) choices for 1st arrow

next edge is harder...

Start at 4: 3 choices
start at 1, 2, 3, 4, : 3 choices
start at 5: 3 choices

] 5×3 ⟹ n(n-2) choices for 2nd arrow

Induction step:
kth arrow



At the kth step: (k-1 arrows chosen already)

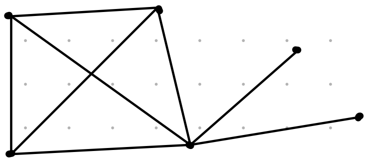
have a rooted forrest of 3 [n-(k-1)] (*) trees. chosing where to end the kth arrows. can't end in own tree, only at the root of a different tree

- pick start: n choices
- end vertex: root of a different tree. there are $n-(k-1) = n-k+1$ but one less: $\underline{n-k}$

(*) Each time we add an edge, we reduce the number of connected components. Started with n connected components [n vertices] & each time we add an edge, take one away, so added k-1 edges so n-(k-1) trees at kth step.

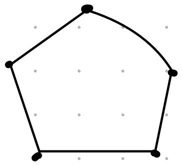
$$\Rightarrow |X| = (n(n-1))(n(n-2))\dots n(n-k) = n^{n-1}(n-1)! = \underbrace{T_n \cdot n \cdot (n-1)!}_{\text{from way 1}} \Rightarrow T_n = n^{n-2}$$ ■

Observation: If G contains K_r as a subgraph [can find k of the vertices so they're all adjacent, then $\chi(G) \geq r$



⇒ ask the reverse Q: if $\chi(G)=r$, then does G contain K_r?

No: G =



$\chi(G) = 3$ but no ▽

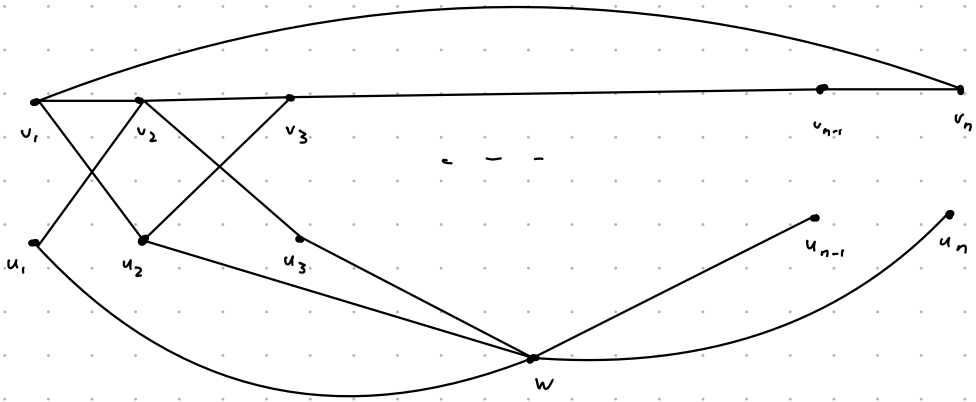
$\boxed{\text{Thm: For any } r \text{ there is a graph } M_r \text{ with } \chi(G)=r \text{ and no triangles (no } K_3)}$

Pf: By construction; we define M_r recursively.

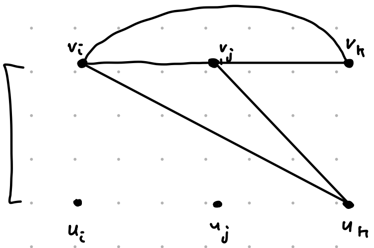
$M_2 =$

$\chi(G)=2$, no triangles.

Given M_{r-1}, we call the vertices of M_{r-1} $v_1, \dots, v_n$. [draw in a line]



- no triangles in the v s ⇒ by induction can't have 3 vs
- no edges between us ⇒ triangle can't contain two u_i
- triangle can't contain w as u, w, u but us not connected. [would need to contain two u_i but u_i not connected.
- only remaining case is v_i, v_j, u_k with edges between, with i, j, k distinct.



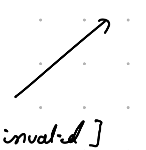
worried about this

← by construction, v_i connected to v_k, v_j connected to u_1, if u_k connected to v_i & v_j.

Now, $\boxed{\text{claim } \chi(M_r) = r}$

Pf: induction

no problems because means



... no problems

M_2

$\Rightarrow \chi(M_2) = 2$

To r-colour M_r:

- First (r-1) colour M_{r-1} [can do by induction] v_i coloured
- then colour u_i same colour as v_i [need to check nothing invalid]
- colour w the last colour.

Need to show there's no valid r-1 colouring.
contradiction & induction

p.t.o.

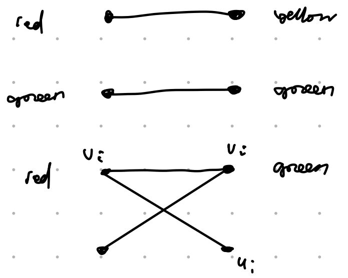
Suppose there was a valid $(r-1)$ colouring of M_r. Suppose that w has colour green.

$\Rightarrow$ u_i is not green for all i.

$\Rightarrow$ recolour M_{r-1} [all the v_i in M_r]: If v_i is green, then change its colour to that of u_i.

↳ this colouring doesn't have green in it.

If the result is a valid colouring then we found a valid $(r-2)$ colouring of M_r ⨯. contradicts induction hypothesis.



fine

never happens

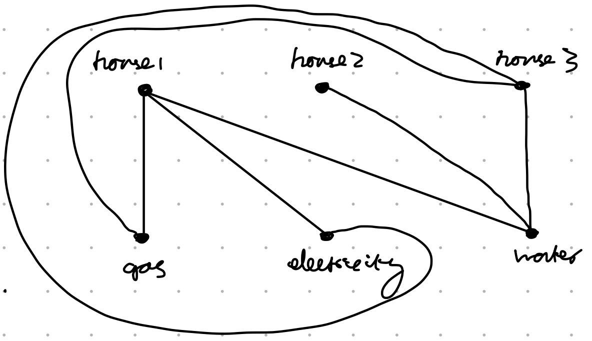
] fine since u_i is not red.

Recall, G has no-odd cycles $\Rightarrow$ G is 2-colourable.

Thm: (Erdős) For any $k, r > 0$, there exist graphs G with chromatic number $\chi(G) > r$ and no cycles of length less than k.

Def: A graph G is planar if it can be drawn in the plane $\mathbb{R}^2$ without edges crossing

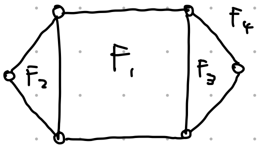
Is $K_{3,3}$ planar? not clear. Have to draw all the possible arrangements. not combinatorial...

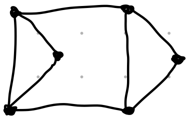


damn!

[can't do it!]

Given a drawing of a planar graph, its faces are the connected components of $\mathbb{R}^2 / G$.



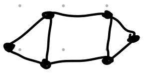


Sketch, thorough in L.N.

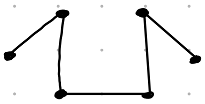
Suppose we have a drawing of a $_\wedge$connected planar graph with $n = |V|$, $e = |E|$, $f = \#$ faces

Thm: Euler's Formula $n - e + f = 2$

Pf:

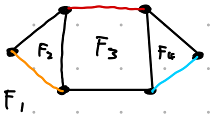


delete edges till spanning tree.



By lemma, drawings of trees have one face. And formula reduces to one that holds $n - e = 1$

Add back in



So we divide face in two. Edges ↑, n same $-e+f$ stays the same.

Started for spanning tree & each time we add an edge, we also add a face so $f-e$ doesn't changed.

Corollary: A simple (no double edges) planar graph with $n \geq 3$ vertices has at most $3n-6$ edges.

Pf: Every face of a planar drawing of G has at least 3 sides.

Each edge contributes to exactly 2 sides, so $\underset{\text{faces}}{3f} \leq \underset{\text{edges.}}{2e}$

Using $2 = n-e+f \Rightarrow 2 \leq n-e+\frac{2}{3}e = n-\frac{e}{3}$

$\Rightarrow 6 \leq 3n-e \Rightarrow e \leq 3n-6$ ■

Corollary: If G is planar, G has a vertex of degree at most 5

Pf:

?

■

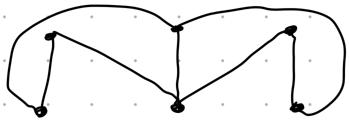
Note: we only get equality if ... and only if every face has exactly 3 sides (is a triangle) [falls out of proof]

Corollary: K_5 is not planar

Pf: $n=5$, $e=10$, $\Rightarrow 10 \not\leq 3 \times 5 - 6 = 9$, K_5 is not planar.

Corollary: $K_{3,3}$ is not planar

3 houses, 3 utilities, can you connect?



etc...

Pf: $K_{3,3}$ has $n=6$, $e=9$. By Euler, if $K_{3,3}$ is planar, then $K_{3,3}$ has 5 faces.
$[6-9+f=2 \Rightarrow f=5]$

This is a bipartite graph, so every cycle goes top, bottom, top, bottom etc. so as $K_{3,3}$ is bipartite, every face has at least 4 sides.

$$4f \leq 2e$$

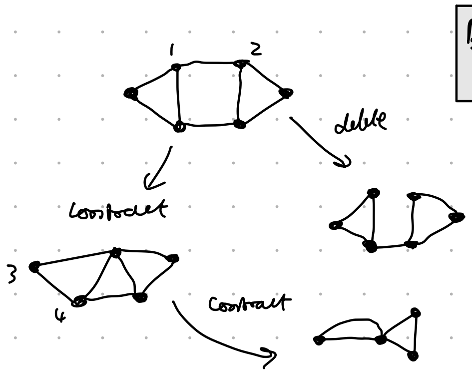
But $20 \not\leq 18 \Rightarrow K_{3,3}$ is not planar.

What ideas should I steal from proofs to use in my own questions?

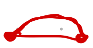
Def: Given a graph G & an edge e of G, we can contract G:
① remove e & merge endpoints.
② delete e.

Def: Given a graph G, a vertex v can be deleted (remove v & all elements connecting it)

Def: a graph H is a minor of a graph G if it can be obtained from G by repeated edge deletion.

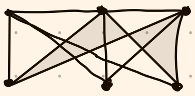


note: don't use to prove a thm about simple graphs. Double contraction
⇓



← bad!

" K_5 and $K_{3,3}$ are the main obstructions to a graph being planar."



can use the fact that $K_{3,3}$ and K_5 are not planar to show more graphs not planar.

Note: G a planar graph, H a minor of G, then H is planar.

⟺

If G has a minor that is not planar, then G is not planar.

⇓

In particular, if G has K_5 or $K_{3,3}$ as a minor, then G is not planar.

Thm: A graph is planar iff it does not have a K_3 or $K_{3,3}$ minor

This is enough!

WTP: If G has no K_5 or $K_{3,3}$ minor, then it is planar.

Pf: By induction on n [number of vertices]

Thm: A graph is planar iff it does not have a K_5 or $K_{3,3}$ minor

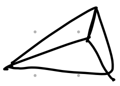
SKETCH PROOF

WTP: If G has no K_5 or $K_{3,3}$ minor, then it is planar

Pf: By induction on n [number of vertices].

Base case: $n=4$. Every simple graph on 4 vertices is a subgraph of K_4. which is planar

If G is not simple, ________



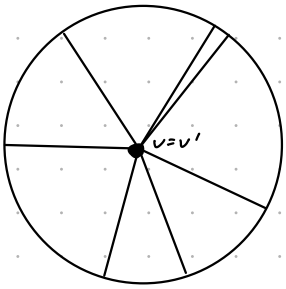
Now suppose Thm holds for vertices $n-1$. G has n vertices & no K_5 or $K_{3,3}$ minor.

Now, pick an edge, $\{v, v'\}$ & contract it to get a new graph G'.

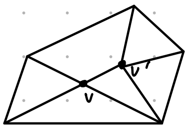
Question: Is G' planar? ⇒ Does

G' has no K_5 or $K_{3,3}$ minor as they would also be minors of G.
G' also has $n-1$ vertices. By induction, G' is planar. Draw it.

"Near" $v'=v$, the picture is like:

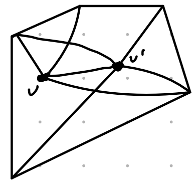


To draw G, we want to split v and v'.

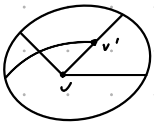


But could have

not planar!!!

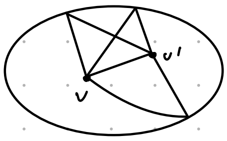


claim: These are the only problems when I split up v and v'.



⇓

$K_{3,3}$ minor



⇓

$K_{5,5}$ minor

■

The 4/5/6 colouring theorem

We will prove the 5/6 colour theorem.

$\underbrace{\chi(G)}_{}\leq 4$ [If G is planar]

HARD

the minimum number of colours needed to colour the vertices of G so that adjacent vertices have different colours

Lets prove the 6 colour Thm

1/12/22

Thm: If G is planar, $\chi(G)\leq 6$

Pf: Assume G is simple & connected [why?]

[connected: if separate, question reduces to connected]

[Simple: can delete all extra edges]. Induct on the number of vertices with base case $n=1$.

Pf: See lecture on 1/12/20

Thm: If G is planar, $\chi(G) \leq 5$

Pf: Assume G simple & connected. Induct on $n = \#$ vertices of G. Base case $n=1$.

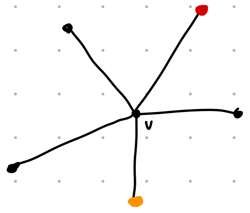
Assume Thm true for graphs with fewer than n vertices, now take G planar with n vertices.

Since G is planar, it has a vertex v of degree at most 5.

Delete v & by induction, 5 colour the resulting graph G'. [n-1 vertices in G']

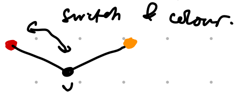
If the neighbours of v have at most 4 colours, then there is a colour left for v [easy case, pg 64]

So assume the neighbours have 5 colours.

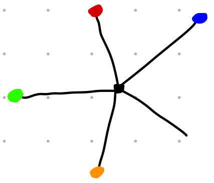


Consider the subgraph of vertices coloured red & yellow and edges between them.

If the red & yellow neighbours of G live in different connected components of the subgraph, we can switch red & yellow of one component to get no neighbour of v red so can colour v red.



Otherwise, there is a red-yellow path connecting the red & yellow neighbours of v.



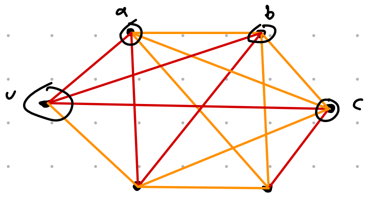
Either blue/green neighbours are in different connected components so can switch & have colour left for v

Or there is a blue-green path from blue neighbour to green neighbour.

Red-yellow, blue-green paths must cross ⇒ impossible as G is planar.

Idea: draw paths, can't cross ⇒ 5 colours.

Question: If 6 ppl in a room, can you find 3 ppl who all know each other or 3 who all don't know each other.

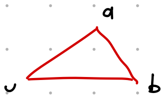


Ramsey Theory

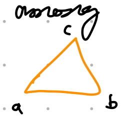
Yes!

Proof: Choose a vertex v. There exist 3 red or 3 yellow edges from v. Call them a, b, c. WLOG, suppose 3 red edges from v. Call the endpoints a, b, c. Consider edges between a, b, c.

If red edge among a, b, c then triangle



If not ∃



∎

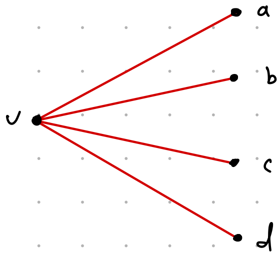
Given a 2-edge colouring of K_{10}, can I find a red K_3 or a yellow K_4?
[10 ppl in a room, 3 ppl know each other or 4 who don't, can you find it?]

Pf: Each vertex has degree 9. choose v.

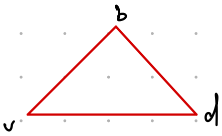
From v there exist either 4 red or 6 yellow edges.

Case 1

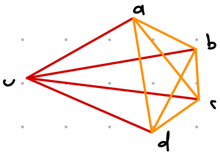
First, suppose there are 4 yellow edges.



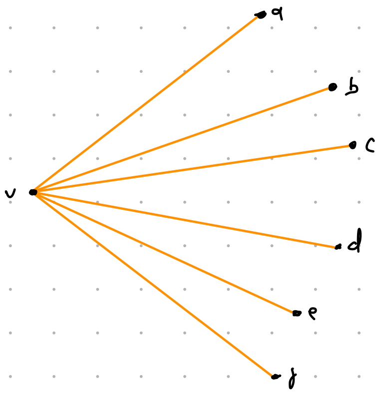
If ∃ red edge among a, b, c, d, then we have a red triangle



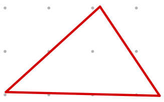
then all edges between a, b, c, d are yellow, so we have a yellow K_4, with vertices a, b, c, d.



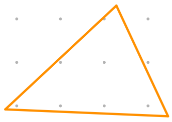
Case 2



Among a, b, c, d, e, f can find either

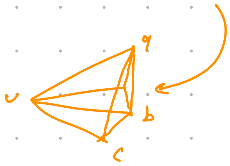


or



↑ Immediately done

↓ add v & get a yellow K_4



done ✓

∎

Thm (Ramsey): For any $k, l > 0$, I can find a number $R(k,l)$ s.t. every 2 edge colouring of $K_{R(k,l)}$ has a red K_k or a yellow K_l.

$R(k,l)$ is the minimum such number.

Shown $R(3,3) \leq 6$, $R(3,4) \leq 10$.

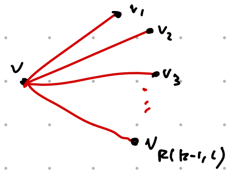
Pf: Work in $K_{R(k-1,l)+R(k,l-1)}$

Choose v. From v, there are $R(k-1,l)+R(k,l-1)$ edges.

Can find either $R(k-1,l)$ red or $R(k,l-1)$ yellow edges.

[If not, total # edges from v would be $\leq R(k-1,l)-1+R(k,l-1)-1$ $\divideontimes$]
$< \deg(v)$

case 1:

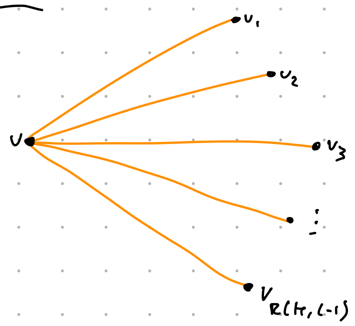


can find a red K_{k-1} or yellow K_l

add v & get red K_k

done

case 2:



Among v_i, can find a red K_k [done] or a yellow K_{l-1}, then add v to get a yellow K_l. [done]

■

Q: What structures always show up in sufficiently large graphs?

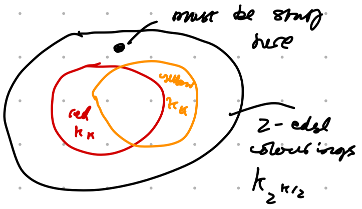
$R(k,l) \leq R(k-1,l)+R(k,l-1)$

Induction proof: $R(k,l) \leq \binom{k+l-2}{k-1}$ ← using proof of theorem, we can find an upper bound. lower bounds are hard....

Lower Bounds for $R(k,l)$ [using random graph theory]

Thm: $R(k,k) \geq 2^{k/2}$

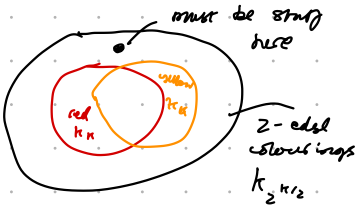
Pf:



Goal: if $n \leq 2^{k/2}$, then the # 2-edge colourings of K_n with red $K_k \leq \frac{1}{2} 2^{n/2}$

Thm: $R(k,k) \geq 2^{k/2}$

Pf:



Goal: if $n \leq 2^{k/2}$, then the # 2-edge colourings of K_n with red $K_k \leq \frac{1}{2} 2^{n/2}$

For each choice of k vertices

Snap thinking: From the possible K_k, how many graphs are they in?

choosing a K_k graph from n

Count $\sum_{\substack{\text{2 edge colourings}\\ \text{of } K_n}} \#\text{ red } K_k$

\# edges K_n

remaining edges to choose from?

For k chosen vertices, they form a red K_k in $2^{\binom{n}{2}-\binom{k}{2}}$ edge colouring

Total : $\boxed{\sum_{\text{colourings}} \#\text{ red } K_k} = \binom{n}{k} \cdot 2^{\binom{n}{2}-\binom{k}{2}}$

\# 2 edge colourings with <u>at least</u> one red K_k is $\leq$ that sum since each contributes ≥ 1 to the sum.

we want this

$$\sum_{\text{colourings}} \#\text{ red } K_k = 0+0+2+k+0+0+1+\dots$$

if $n \leq 2^{k/2}$

$$\left(\begin{array}{l}\text{The number of 2 edge}\\ \text{colourings with a red } K_k\end{array}\right) \leq \binom{n}{k} \cdot 2^{\binom{n}{2}-\binom{k}{2}} \underset{\text{claim}}{<} \frac{1}{2} \cdot 2^{\binom{n}{2}}$$

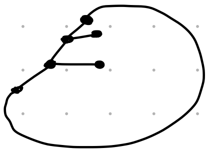
$$\begin{aligned}
\binom{n}{k} \cdot 2^{\binom{n}{2} \cdot \binom{k}{2}} &= \frac{n \cdot (n-1) \cdots (n-k+1)}{k \cdot (k-1) \cdots 2 \cdot 1} \cdot 2^{\binom{n}{2} \cdot \binom{k}{2}} \\
&< \frac{n^k}{k \cdot (k-1) \cdots 2 \cdot 1} \cdot 2^{\binom{n}{2}-\binom{k}{2}} && k! < 2^k \\
&< \frac{n^k}{2^k} \cdot 2^{\binom{n}{2}-\binom{k}{2}} && n \leq 2^{k/2} \\
&\leq \frac{2^{k^2/2}}{2^k} \cdot 2^{\binom{n}{2}-\binom{k}{2}} \\
&= 2^{\frac{k^2}{2}-k-\binom{k}{2}} \cdot 2^{\binom{n}{2}} \\
&= 2^{\frac{k^2-2k}{2}-\frac{k(k-1)}{2}} \cdot 2^{\binom{n}{2}} \\
&= \frac{1}{2^{k/2}} \cdot 2^{\binom{n}{2}} \\
&\leq \frac{1}{2} \cdot 2^{\binom{n}{2}}
\end{aligned}$$

$\Rightarrow$ there exists a 2-edge colouring with no red K_k or yellow K_k

$\Rightarrow R(k,k) > 2^{k/2}$

Thm: Fix $k>0$. There exist graphs with no cycles of length $\leq k$ with arbitrarily high chromatic number.

Pf: Fix $k>0$, and fix $r>0$. Want no short cycles (length less than k), but chromatic number r large.



For n very large, pick a graph G by taking n vertices & for each pair, add an edge with probability $p = \frac{1}{n^{1-\varepsilon}}$ for $0<\varepsilon<<1$ [ε very small]

(ε will be $\frac{1}{k}$ good enough)

claim 1: On average, this graph created G has at most $\frac{1}{2}(k-2)n^{k\varepsilon}$ short cycles.

For each candidate cycle (list of k vertices $v_1, \dots, v_k$), this contributes to the average with probability p^k. [probability that k vertices give a k-cycle]

n choices for 1st vertex, n-1 for 2nd

$$E_k = \left(\begin{matrix}\text{the average number}\\ \text{of } k \text{ cycles}\end{matrix}\right) = \frac{n(n-1)\dots(n-k+1)}{k \cdot 2} \times p^k$$] expected number of k-cycles.

don't care about order

can reverse direction

times probability that that k-cycle exists

$$\leq \frac{n^k}{2k} \cdot p^k$$

$$\leq \frac{n^k p^k}{2} \quad \text{[course-enough estimate]}$$

But should add up $E_3, E_4, \dots, E_k$ [discount E_1 & E_2 - not cycles]

$$E_3 + E_4 + \dots + E_k < \sum_{i=3}^{k} \frac{n^i p^i}{2} = \sum_{i=3}^{k} \frac{1}{2} n^{i\varepsilon} \leq \sum_{i=3}^{k} \frac{1}{2} n^{k\varepsilon} = \frac{1}{2}(k-2)n^{k-\varepsilon}$$

HHHAAA

claim 2: If n is large, G is unlikely to have more than $\frac{n}{2}$ short cycles.

Markov's Inequality: $\mathbb{P}$(at least $\frac{n}{2}$ short cycles),

[in worst case, where every graph has $\frac{n}{2}$ or zero short cycles] $\quad E = \frac{n}{2}p + 0(1-p)$

Markov: $p \leq \frac{2}{n} \cdot E < (k-2)n^{k\varepsilon - 1}$ $\qquad k\varepsilon - 1 < 0$

so as $n \to \infty$, $p \to 0$ so G unlikely to have short cycles.

[if you decrease p s.t. **no** short cycles, will not have high chromatic number.]

claim 3: If n large, then with high probability, G will satisfy

$$\text{ind}_v(G) < \frac{n}{2r}$$

Why enough? Pick our G, it will have no more than $\frac{n}{2}$ short cycles. Delete a vertex from each cycle. Call the resulting graph H. H now has no short cycles. Want H to have high chromatic number, $\chi(H) \geq r$. As each colour is in an independent set, its enough to say

$$\text{ind}_v(H) < \frac{\# V(H)}{r} \quad \text{(since each colour class is an independent set)}$$

Then, can't fill out all vertices with r independent sets.

will imply that $\chi(H) > r$

So as at least $\frac{n}{2}$ vertices left in H,

$$\text{ind}_v(H) < \frac{n/2}{r} \left(\leq \frac{\#V(H)}{r} \right)$$

Q: $\text{ind}_v(G) \geq \text{ind}_v(H)$? H created only by deleting vertices.

True $\because$ we got H by deleting vertices $\Rightarrow$ vertex independence number can't go up.

Note: proof before, red edge or yellow edge. Now edge or no edge. Same as finding a yellow K_k. very similar.

Pf of claim 3: For any given $\frac{n}{2r}$ vertices, probability that they form an independent set in G. There are $\binom{\frac{n}{2r}}{2}$ edges, for edge to not exist $(1-p)$.

$$\longrightarrow (1-p)^{\binom{\frac{n}{2r}}{2}}$$

probability that G has an independent set of size $\frac{n}{2r}$

Expected # of such independent sets is $= \binom{n}{\frac{n}{2r}} \cdot (1-p)^{\binom{\frac{n}{2r}}{2}}$

[choosing from n, $\frac{n}{2r}$ vertices]

Worst case scenario: $E = 1 \cdot \text{Prob(independent set)} + 0(1 - \text{Prob(Indepent set)})$

$$\text{Markov} \Rightarrow \quad \text{Prob(Indpt set)} \leq \binom{n}{\frac{n}{2r}} \cdot (1-p)^{\binom{n/2r}{2}}$$

$$= \binom{n}{\frac{n}{2r}} \cdot \left(1 - \frac{1}{n^{1-\varepsilon}}\right)^{\binom{n/2r}{2}} \xrightarrow[n \to \infty]{[\text{work}]} 0$$

Small # independent sets $\Rightarrow$ high chromatic number.

$$\boxed{\text{WTS: } \lim_{n \to \infty} \binom{n}{\frac{n}{2r}} \cdot \left(1 - \frac{1}{n^{1-\varepsilon}}\right)^{\binom{n/2r}{2}} = 0}$$

[see notes]