MA133 Review — Differential Equations — 23/5/22

- **Autonomous**: the independent variable doesn't appear in the ODE.
  on bottom of derivative

- **Linear**: the dependent variable appears only in terms of its derivatives. E.g. no $\ddot{x} + x^2 = 0$ allowed as $\because$ x^2 term makes it non-linear.

- **FTC**: Suppose $f : [a,b] \longrightarrow \mathbb{R}$ is continuous and for $a \le x \le b$, define

$$G(x) = \int_a^x f(\tilde{x})\, d\tilde{x}$$

then

$$\frac{dG}{dx} = f(x)$$

Furthermore,

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

$\forall F$ satisfying $F'(x) = f(x)$

- **N2** $\Delta p = F\Delta t \implies \frac{d}{dt}(mv) = F(t) \implies F = ma$

- **Solution** Given an open interval I that contains t_0, a solution to the IVP

$$\frac{dx}{dt} = f(t) \quad : \quad x(t_0) = x_0$$

on I is a continuous function $x(t)$ with $x(t_0) = x_0$ and

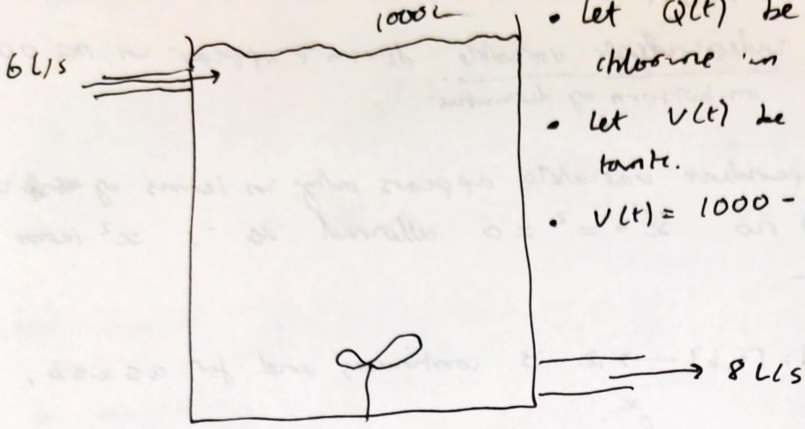
$$x'(t) = f(t). \qquad \forall t \in I$$

- **Existence & Uniqueness** If $f(x,t)$ and $\frac{\partial f}{\partial x}$ are continuous for $a < x < b$ and $c < t < d$, then $\forall x_0 \in (a,b)$ and $t_0 \in (c,d)$, the IVP problem

$$\frac{dx}{dt} = f(t) \quad : \quad x(t_0) = x_0$$

has ~~solution~~ a unique solution on some open interval $I \ni t_0$.

Simple Mixing Problem



- let $Q(t)$ be the amount of chlorine in the tank
- let $V(t)$ be the volume of the tank.
- $V(t) = 1000 - 2t$ at t seconds.

- How to measure concentration?

$$\frac{dQ}{dt} = \text{rate of change of chlorine}$$

$$\frac{Q(t)}{V(t)} = \text{concentration of chlorine in the tank.}$$

$$8\,\frac{Q(t)}{V(t)} = \text{how much chorine is pumped out per second.}$$

$$\frac{dQ}{dt} + 8\cdot\frac{Q(t)}{V(t)} = 0$$

$$\Rightarrow \quad \frac{dQ}{dt} + \frac{8\,Q(t)}{1000-2t} = 0$$

$$I = e^{\int \frac{8}{1000-2t}\,dt} = e^{\int \frac{4}{500-t}\,dt} = e^{\cancel{4}(\ln|500-t|)^{-4}} = \frac{1}{(500-t)^4}$$

$$\therefore \quad \frac{1}{(500-t)^4}\frac{dQ}{dt} + \frac{4}{500-t}\cdot Q(t)\cdot\frac{1}{(500-t)^4} = 0$$

$$\Rightarrow \quad \frac{1}{(500-t)^4}\frac{dQ}{dt} + \frac{4}{(500-t)^5}\,Q = 0$$

$$\frac{d}{dt}\left(\frac{1}{(500-t)^4}\,Q\right) = 0 \qquad\qquad 0.02 = \frac{Q}{1000}$$

$$\frac{1}{(500-t)^4}\,Q(t) = C \qquad\qquad 20 = Q$$

$$Q(t) = C(500-t)^4$$

$$20 = C(500)^4 \Rightarrow C = \frac{20}{500^4}$$

$$\therefore \quad Q(t) = 20\left(\frac{500-t}{500}\right)^4 \qquad 0 \le t \le 500.$$

Integrating Factor

$$\frac{dy}{dt} + r(t)y = g(t)$$

Multiply by $\cancel{e^{\int r(t)dt}}$ $I(t)$ [some function of t]

$$I(t)\frac{dy}{dt} + r(t)I(t)y = g(t)I(t) \quad (*)$$

we know $\frac{d}{dt}(I(t)y(t)) = I(t)\frac{dy}{dt} + I'(t)y$

so comparing with $(*)$, $r(t)I(t) = I'(t)$

this is a differential equation, $\frac{dI}{dt} = r(t)I(t)$ with solution

$$I(t) = e^{\int r(t)dt}$$

$$\therefore \quad e^{\int r(t)dt}\frac{dy}{dt} + r(t)e^{\int r(t)dt}y = g(t)e^{\int r(t)dt}$$

$$\frac{d}{dt}\left(e^{\int r(t)dt}y\right) = g(t)e^{\int r(t)dt}$$

[separation of variables too!]

$$\Longrightarrow \quad e^{\int r(t)dt}y = \int g(t)e^{\int r(t)dt}dt + A$$

so $\cancel{y = e^{t}\int e^{\int r(t)dt} g(t)\,dt + A}$

$$\boxed{y = e^{-\int r(t)dt}\int e^{\int r(t)dt}g(t)\,dt + Ae^{-\int r(t)dt}}$$

Population Dynamics

$$\frac{dN}{dt} = rN\left(1 - \frac{N}{K}\right)$$

Substitution Methods

Example 1.17 : $xy + y^2 + x^2 - x^2 \frac{dy}{dx} = 0$

let $u = \frac{y}{x} \Rightarrow y = ux \quad \frac{dy}{dx} = \frac{du}{dx} x + u$

so substituting

$$x(ux) + (ux)^2 + x^2 - x^2\left(\frac{du}{dx} x + u\right) = 0$$

$$\cancel{ux^2} + u^2x^2 + x^2 - \frac{du}{dx}x^3 - \cancel{ux^2} = 0$$

so for $x \neq 0$, $\frac{du}{dx} x^3 = x^2(1+u^2)$

$$\frac{du}{dx} x = 1 + u^2$$

$$\Rightarrow \int \frac{1}{1+u^2}\, du = \int \frac{1}{x}\, dx$$

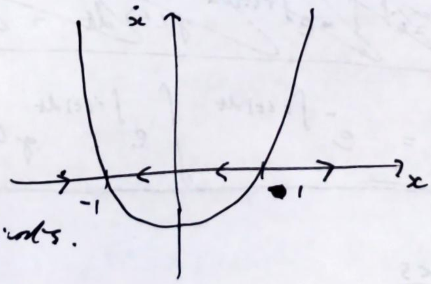
$$\tan u = \log|x| + c$$

$$u = \arctan(\log|x| + c)$$

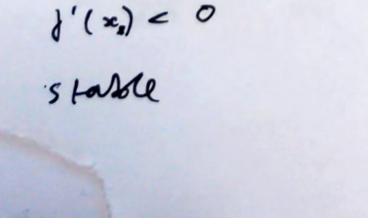
$$y = x \arctan(\log|x| + c)$$

$\boxed{\text{Fixed Points}}$

$$\frac{dx}{dt} = f(x)$$

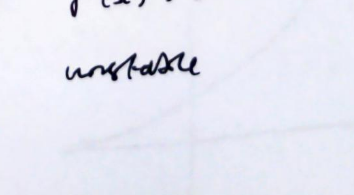


$\frac{dx}{dt} = 0 \Rightarrow$ stationary points.



$f'(x_0) < 0$

stable



$f'(x) > 0$

unstable

$$\boxed{a\frac{d^2x}{dt^2} + b\frac{dx}{dt} + cx = 0}$$

$\lambda_{\pm} = p \pm iq$

① real roots

$x(t) = Ae^{\lambda_1 t} + Be^{\lambda_2 t}$

② Repeated roots

$x(t) = (A + Bt)e^{\lambda t}$

③ complex roots

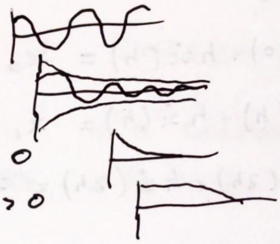
~~$x(t) = A\cos\lambda_1 t + B\sin\lambda_2 t$~~

$x(t) = e^{pt}(A\cos qt + B\sin qt)$

Mass spring systems

$$m\frac{d^2x}{dt^2} + c\frac{dx}{dt} + kx = 0$$

- undamped: $c = 0$
- underdamped: $c^2 - 4mk < 0$
- critically damped: $c^2 - 4mk = 0$
- overdamped: $c^2 - 4mk > 0$

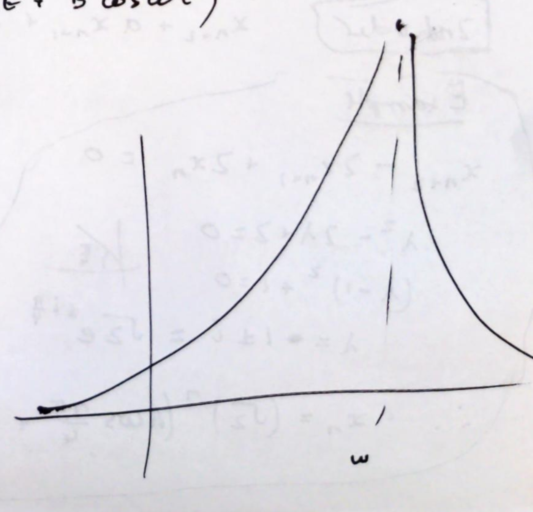


Inhomogeneous case, $f(t)$

$f(t)$	Trial solution
ae^{kt} [k not root of aux]	Ae^{kt}
ae^{kt} [k is a root]	Ate^{kt} or At^2e^{kt} if repeated root
$a\sin\omega t$ / $a\cos\omega t$	$A\sin(\omega t) + B\cos(\omega t)$
at^n	$P(t)$, general polynomial, degree n
$at^n e^{kt}$	$P(t)e^{kt}$ [P(t) defined ↑]
$t^n(a\sin(\omega t) + b\cos(\omega t))$	$P_1(t)\sin(\omega t) + P_2(t)\cos(\omega t)$
$e^{kt}(a\sin\omega t + b\cos\omega t)$	$e^{kt}(A\sin\omega t + B\cos\omega t)$

With Forcing

$$m\frac{d^2x}{dt^2} + c\frac{dx}{dt} + \omega^2 x = F\cos\Omega t$$



Chapter 3

Euler's Method

ODE given $\frac{dx}{dt} = f(t, x)$: $x(0) = x_0$

solution $x(t)$. lets quantise time, so $t = nh$

Note $x(t+h) = x(t) + h\dot{x}(t) = x(t) + hf(t, x)$

so lets start at $t=0$. want to get $x_{n+1} = g(x_n)$

$x(0) = x_0$

$x(h) = x(0) + h\dot{x}(h) = x_0 + hf(0, x_0)$

$x(h+h) = x(h) + h\dot{x}(h) = x_1 + hf(h, x_1)$

$x(2h+h) = x(2h) + h\dot{x}(2h) = x_2 + hf(2h, x_2)$

$\vdots$

$x_{k+1} = x(k) + h\dot{x}(k) = x_k + hf(hk, x_k)$

$\Rightarrow$ $\cancel{x_{n+1} = x_n + h f(nh, x_k)}$

$$\boxed{x_{k+1} = x_k + hf(kh, x_k)}$$

Solving Difference Equations

$x_{n+1} = f(x_n, n)$

Linear $x_{n+1} = a x_n \Rightarrow x_n = a^n x_0$

2nd order $x_{n+2} + a x_{n+1} + b x_n = 0 \Rightarrow$

- distinct roots: $x_n = Ak_1^n + Bk_2^n$
- repeated roots: $x_n = Ak^n + Bnk^n$
- complex roots $k = p \pm iq$: $x_n = r^n(A\cos n\theta + B\sin n\theta)$, $k = re^{\pm i\theta}$

Example

$x_{n+2} - 2x_{n+1} + 2x_n = 0$

$\lambda^2 - 2\lambda + 2 = 0$

$(\lambda - 1)^2 + 1 = 0$

$\lambda = 1 \pm i = \sqrt{2}e^{\pm i\frac{\pi}{4}}$

$\therefore x_n = (\sqrt{2})^n \left(A\cos\frac{n\pi}{4} + B\sin\frac{n\pi}{4}\right)$

Non Homogeneous

$f_n(t) =$ polynomial $\rightarrow$ try poly in n.

Obvious guesses, just be careful with repeated roots & stick in another n or n^2.

nk^{an} / $n^2 k^{n^2}$

Fixed pts

Difference Eqn: $x_{n+1} = f(x_n)$

$f(x_n) = x_n \quad \forall n$

Stable: $|f'(x_*)| < 1$, unstable $|f'(x_*)| > 1$

Period two orbit: $f(f(x_*)) = x_* \iff f^2(x_*) = x_*$

$\boxed{\text{chapter 4}}$

$\boxed{\text{solution}}$ to the IVP

$$\frac{d}{dx}(\underline{x}(t)) = \underline{f}(\underline{x}, t) \quad : \quad \underline{x}(t_0) = \underline{x}_0 \qquad \underline{x} \in \mathbb{R}^n$$

on the open interval $I \ni t_0$ is a continous function ~~x~~ $\underline{x} : I \longrightarrow \mathbb{R}^n$ with $\underline{x}(t_0) = \underline{x}_0$ and $\underline{\dot{x}}(x, t) = \underline{f}(\underline{x}, t)$.

$\boxed{\text{Jacobian Matrix}}$

$$D\underline{f} = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial x_1} & \cdots & \frac{\partial f_n}{\partial x_n} \end{pmatrix}$$

$\boxed{\text{Existance \& Uniqueness}}$. If $\underline{f}(\underline{x}, t)$ and $D\underline{f}(x, t)$ are continous for $\underline{x} \in U \subseteq \mathbb{R}^n$, $a \le t \le b$, then for any $\underline{x}_0 \in U$, $t_0 \in (a, b)$ $\exists$ a unique solution to

$$\frac{d}{dx}(\underline{x}(t)) = f(\underline{x}, t) \quad : \quad \underline{x}(t_0) = \underline{x}_0$$

$$\left.\begin{aligned} \frac{dx}{dt} &= px + qy \\ \frac{dy}{dt} &= rx + sy \end{aligned}\right. \implies \underline{\dot{x}} = \begin{pmatrix} p & q \\ r & s \end{pmatrix} \underline{x} = A\underline{x}$$

Try solution $\underline{x}(t) = e^{\lambda t} \underline{v}$

$\implies \lambda e^{\lambda t} \underline{v} = A e^{\lambda t} \underline{v}$

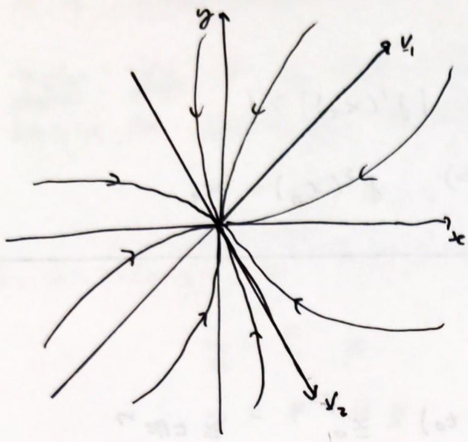
$\implies A\underline{v} = \lambda \underline{v}$

$\implies \underline{v}$ are eigenvectors of A

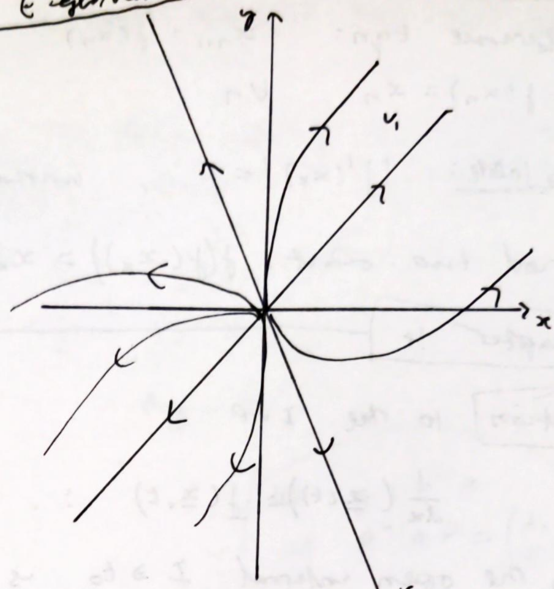
λ are eigenvalues.

$\boxed{\dot{\underline{x}} = A\underline{x}}$

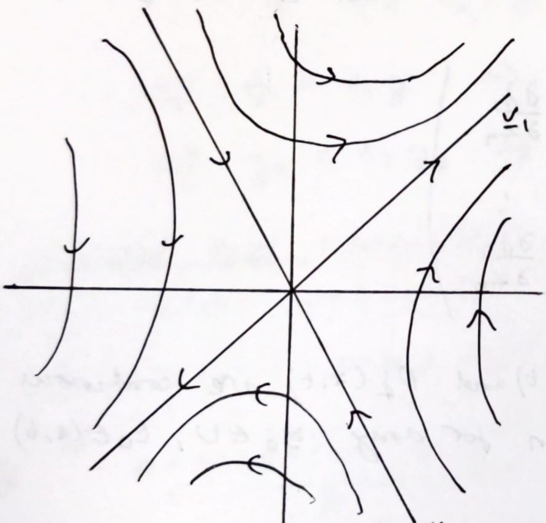
Real Eigenvalues



$\lambda_1 < \lambda_2 < 0$

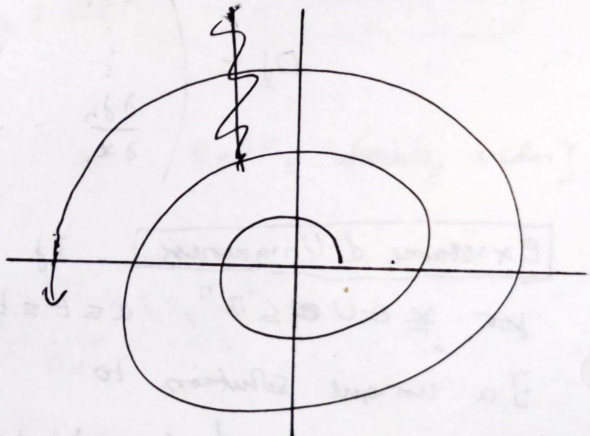


$\lambda_1 > \lambda_2 > 0$

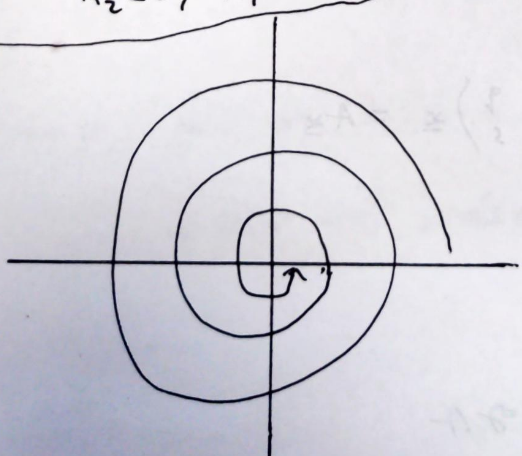


$\lambda_2 < 0, \quad \lambda_1 > 0$

Complex Eigenvalues



$Re(\lambda) > 0$ [unstable]



$Re(\lambda) < 0$

Diagonalisation / Uncoupling

Consider $\dot{\underline{x}} = A\underline{x}$ where A has distinct eigenvalues [so A is diagonalisable].

<u>Note</u>: ~~If~~ P is the change of base matrix from standard basis to basis of eigenvectors, then $[P = (\underline{v}_1 | \underline{v}_2)]$

$$P^{-1} A P = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$

so lets change the coordinates. let $y = P^{-1}\underline{x}$

$P y = \underline{x}$

$$\therefore \quad \dot{y} = P^{-1}\dot{x}$$

$$\dot{y} = P^{-1} A \underline{x}$$

$$\dot{y} = P^{-1} A P y$$

$$\dot{y} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} y$$

let $y = P^{-1} A$
as $P^{-1} A P = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$

$$\Rightarrow \dot{y}_1 = \lambda_1 y_1 \Rightarrow y_1 = Ae^{\lambda_1 t}$$
$$\Rightarrow \dot{y}_2 = \lambda_2 y_2 \Rightarrow y_2 = Be^{\lambda_2 t}$$

$\boxed{\text{uncoupled equations!}}$

Repeated Real Eigenvalues

Try: $\underline{x}(t) = (\underline{a} + \underline{b}t)e^{\lambda t}$ [we know $Be^{\lambda t}\underline{v}$ already a soln]

$$\dot{\underline{x}}(t) = \underline{b}e^{\lambda t} + \lambda(\underline{a} + \underline{b}t)e^{\lambda t}$$

so $$\underbrace{\lambda \underline{a}e^{\lambda t} + \underline{b}e^{\lambda t} + \lambda \underline{b}te^{\lambda t}}_{\dot{x}} = \underbrace{A\underline{a}e^{\lambda t} + A\underline{b}te^{\lambda t}}_{Ax}$$

Equating coefficients:

$$\lambda \underline{a} + \underline{b} = A\underline{a} \iff (A - \lambda I)\underline{a} = \underline{b}$$
$$\lambda \underline{b} = A\underline{b} \iff (A - \lambda I)\underline{b} = 0 \Rightarrow \underline{b} \text{ eigenvector}$$

Find $\underline{a}$ [put into equations, there are many solutions].

$$\underline{x}(t) = Be^{\lambda t}\underline{v} + C(\underline{a} + \underline{v}t)e^{\lambda t}$$

Complex Eigenvalues

$\dot{\underline{x}} = A\underline{x}$ $\qquad \lambda = p \pm iq$

$\Rightarrow \quad \underline{x}(t) = e^{pt}\left[(a\cos qt + b\sin qt)\underline{v}_1 + (b\cos qt - a\sin qt)\underline{v}_2\right]$

Direction we rotate depends on the sign of q.

Directional Derivative

$D_{\underline{u}}(f(x)) = \nabla f \cdot \underline{u}$

∇f is direction of steepest descent.

$\boxed{\text{Approximating the solutions of a } 2\times 2 \text{ system near fixed pts}}$

$$\frac{dx}{dt} = f_1(x,y) \qquad \qquad \Rightarrow \quad \dot{x} = f(\underline{x})$$
$$\frac{dy}{dt} = f_2(x,y)$$

[Assume a fixed point $f(\underline{x}_*) = \underline{0}$, and consider a small change $\underline{u} = (u, v)$

$\therefore \quad x = x_* + u, \quad y = y_* + v$

$$\therefore \quad \frac{dx}{dt} = \frac{d}{dt}(x_* + u) = f_1(\underline{x}_* + \underline{u}) \simeq f_1(x_*) + \nabla f_1 \cdot \underline{u}$$

$$\Rightarrow \frac{du}{dt} = \nabla f_1 \cdot \underline{u}$$

similarly $\quad \dfrac{dv}{dt} = \nabla f_2 \cdot \underline{u}$

[applied to population model in notes]

$$\therefore \quad \dot{\underline{u}} = \begin{pmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{pmatrix}\Bigg|_{(x_*, y_*)}$$

STEPS

1. Find fixed pts
2. evaluate ∇f at fixed points
3. Find eigenvalues & eigenvectors
4. Phase portrait

$\therefore$ close to fixed pt, solutions behave

$$\dot{\underline{x}} = \nabla f(x_*)\, \underline{x}$$

$\Rightarrow$ stability of fixed pt is given by the eigenvalues of ∇f